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Tian Invariant on the Blown-up Grassmannian Futaki Obstructions and K-Stability

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Abstract

In this paper, we compute the Tian invariant on a generalization of Calabi manifolds introduced by the author in collaboration with A. Ben Abdesselem and M. Filali in [1]. We adapt the method of the original article to the setting of complex Grassmannians blown up along linear subvarieties. Our original contribution is fourfold: we explicitly determine the subgroup $H \subset \mathrm{Aut}(G(k,n)) = \mathbb{P}\,\mathrm{GL}(n,\mathbb{C})$ that preserves the linear subvariety L, and we show that the functions ψ_j are exactly the functions invariant under this group; we establish a new obstruction to the existence of Kähler-Einstein metrics on X in terms of the Tian invariant and the Futaki obstruction; we show that for certain values of k and n, the Tian invariant attains the upper bound $\alpha(X) = \frac{\dim X}{\dim X+1}$, which ensures the existence of a Kähler-Einstein metric; and we establish an equivalence between the K-stability of X and a simple numerical condition on k and n. These results are new and generalize previous work on Calabi manifolds.

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1. Introduction and Preliminaries

The Tian invariant, introduced by G. Tian in [2], is a numerical invariant that measures the stability of a Fano Kähler manifold. It is defined as the supremum of $\alpha > 0$ such that there exists a constant C_α satisfying

$$\int_X e^{-\alpha\phi}\, dV \leq C_\alpha \tag{1}$$

for every admissible function ϕ invariant under a group of automorphisms G, with $\sup_X \phi = 0$. The Tian invariant plays a crucial role in the existence of Kähler-Einstein metrics: if $\alpha(X) > \frac{n}{n+1}$, then there exists a Kähler-Einstein metric on X. This criterion, established by Tian [2], has become a fundamental tool in Kähler geometry, complementing the classical results of Yau [19] and Aubin [4] in the negative and zero cases, respectively. The invariant is closely related to the concept of K-stability introduced by Tian and Donaldson [11], and to the Futaki obstruction [3], which provides a necessary condition for the existence of Kähler-Einstein metrics.

The study of Tian invariants on explicit families of Fano manifolds has attracted considerable attention in recent years. In [1], the authors computed the Tian invariant on a family of generalized Calabi manifolds, defined as submanifolds of $\mathbb{P}_{m-1}\mathbb{C} \times \mathbb{P}_{mn}\mathbb{C}$. These manifolds, introduced by Calabi [5] in the context of extremal Kähler metrics, provide a rich class of examples where explicit computations are possible. The method developed in [1] relies on the construction of extremal functions ψ_j that are invariant under a suitable group of automorphisms, and on a symmetrization argument that allows one to reduce the computation of the Tian invariant to the convergence of certain integrals.

Our goal in this paper is to extend this method to complex Grassmannians blown up along linear subvarieties. The Grassmannian $G(k, n)$ is a Fano manifold with a rich geometry, and its blow-up along a linear subvariety provides a natural generalization of the Calabi manifolds considered in [1]. The resulting varieties, which we call Calabi Grassmannians, exhibit interesting stability properties that depend on the numerical parameters k and n. Our work is also motivated by the recent developments in the theory of K-stability, in particular the Chen-Donaldson-Sun theorem [13], which establishes the equivalence between the existence of Kähler-Einstein metrics and K-stability for Fano manifolds. For more information, see [6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18].

1.1. **Structure of the paper.** The paper is organized as follows. In Section 2, we recall the basic notions about the complex Grassmannian and its automorphism group. In Section 3, we construct the Calabi Grassmannian and the associated Kähler metric. In Section 4, we define the extremal functions ψ_j and establish their properties. Section 5 contains the proof of the main theorem on the Tian invariant. Section 6 is devoted to the computation of the Futaki obstruction.

Section 7 establishes the link between the Tian invariant and K-stability. Section 8 contains detailed examples and a discussion of exceptional cases. Finally, Section 9 presents conclusions and perspectives.

1.2. **Novelties of this paper.** This paper brings the following original contributions:

- **Explicit description of the automorphism group**: We determine the subgroup $H \subset \mathbb{P}\,\mathrm{GL}(n,\mathbb{C})$ that preserves a linear subvariety L of $G(k,n)$. We show that

(2)
$$H = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathbb{P}\,\mathrm{GL}(n,\mathbb{C}) : A \in \mathrm{GL}(k,\mathbb{C}), C \in \mathrm{GL}(n-k,\mathbb{C}), B \in \mathcal{M}_{k,n-k}(\mathbb{C}) \right\}.$$

- **Explicit computation of invariant functions**: We show that the functions ψ_j defined by (**??**) and (2) are exactly the H-invariant and almost psh functions on X. This result is new and generalizes Lemma 1 of [1].
- **Futaki obstruction**: We compute the Futaki obstruction for X and show that it vanishes if and only if $k(n-k) = n-1$. This gives a necessary condition for the existence of a Kähler-Einstein metric.
- **Existence criterion**: We show that for n sufficiently large and k satisfying $k(n-k) < n-1$, the Tian invariant satisfies

$$\alpha(X) = \frac{1}{n-k+1} > \frac{k(n-k)}{k(n-k)+1} = \frac{\dim X}{\dim X + 1}, \tag{3}$$

  which ensures the existence of a Kähler-Einstein metric on X.
- **K-stability**: We establish an equivalence between the K-stability of X and the condition $k(n-k) < n-1$.

1.3. **Fundamental notations.** Throughout this paper, we use the following notations:

- $G(k,n)$: complex Grassmannian of k-planes in $\mathbb{C}^n$.
- L: linear subvariety of $G(k,n)$ of codimension d.
- $X = \mathrm{Bl}_L\, G(k,n)$: Calabi Grassmannian.
- $E = \pi^{-1}(L)$: exceptional divisor.
- H: subgroup of $\mathrm{Aut}(G(k,n))$ preserving L.
- ω_{FS}: Fubini-Study Kähler form.
- g: Kähler metric on X.
- $\alpha(X)$: Tian invariant of X.
- $\mathrm{Fut}(X)$: Futaki obstruction of X.

2. Geometry of the complex Grassmannian

2.1. **The Grassmannian as a projective variety.** Let $G(k,n)$ be the complex Grassmannian of k-dimensional vector subspaces in $\mathbb{C}^n$. It is a smooth projective variety of dimension $k(n-k)$. Recall that $G(k,n)$ can be embedded into $\mathbb{P}(\wedge^k\mathbb{C}^n)$ via the Plücker embedding:

$$\iota : G(k,n) \hookrightarrow \mathbb{P}(\wedge^k\mathbb{C}^n), \quad \iota([v_1,\ldots,v_k]) = [v_1 \wedge \cdots \wedge v_k]. \tag{4}$$

The Plücker coordinates p_I for $I = (i_1,\ldots,i_k)$ are the $k \times k$ minors of an $n \times k$ matrix representing a k-plane. They satisfy the Plücker relations:

$$\sum_{i=1}^{k+1}(-1)^i p_{i_1,\ldots,i_{k-1},j_i} p_{j_1,\ldots,\widehat{j_i},\ldots,j_{k+1}} = 0. \tag{5}$$

2.2. **Fubini-Study metric.** The Fubini-Study metric on $\mathbb{P}(\wedge^k\mathbb{C}^n)$ induces a Kähler metric on $G(k,n)$ whose Kähler form is

$$\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log\left(\sum_I |p_I|^2\right). \tag{6}$$

In local coordinates $z_{i,j}$ for $1 \le i \le k, k+1 \le j \le n$, the metric is written as:

$$g_{FS} = \sum_{i,j=1}^{k}\sum_{l=1}^{n-k} \frac{(1+\|z\|^2)\delta_{ij}\delta_{lm} - \overline{z}_{i,l}z_{j,m}}{(1+\|z\|^2)^2}\, dz_{i,l} \otimes d\overline{z}_{j,m}, \tag{7}$$

where $\|z\|^2 = \sum_{i,j}|z_{i,j}|^2$.

2.3. **Ricci curvature formula.** The Ricci form of ω_{FS} is given by

$$\mathrm{Ric}(\omega_{FS}) = (n-k+1)\omega_{FS}. \tag{8}$$

This shows that $G(k,n)$ is a Fano manifold with $c_1(G(k,n)) = (n-k+1)\omega_{FS}$.

2.4. **Automorphism group.**

Proposition 2.1. *The automorphism group of $G(k,n)$ is*

$$\mathrm{Aut}(G(k,n)) = \mathbb{P}\,\mathrm{GL}(n,\mathbb{C}) = \mathrm{GL}(n,\mathbb{C})/\mathbb{C}^*. \tag{9}$$

An element $A \in \mathrm{GL}(n,\mathbb{C})$ acts on $G(k,n)$ by

$$A \cdot [v_1,\ldots,v_k] = [Av_1,\ldots,Av_k]. \tag{10}$$

In terms of Plücker coordinates, this action is given by

$$p_I \mapsto \det(A_I)p_I, \tag{11}$$

where A_I is the $k \times k$ matrix formed by the columns of A indexed by I.

Proof. Let $A \in \mathrm{GL}(n, \mathbb{C})$. The map $[v_1, \ldots, v_k] \mapsto [Av_1, \ldots, Av_k]$ is well-defined because if $[v_1, \ldots, v_k] = [w_1, \ldots, w_k]$, then there exists $B \in \mathrm{GL}(k, \mathbb{C})$ such that $w_i = \sum_j B_{ij} v_j$. Then $Aw_i = \sum_j B_{ij} Av_j$, so $[Aw_1, \ldots, Aw_k] = [Av_1, \ldots, Av_k]$. Conversely, every automorphism of $G(k, n)$ comes from an automorphism of the projective space $\mathbb{P}(\wedge^k \mathbb{C}^n)$ that preserves the Grassmannian. Chow's theorem guarantees that these automorphisms are induced by linear transformations of $\mathbb{C}^n$. □

2.5. **Subgroup preserving a linear subvariety.**

Proposition 2.2. *Let $L \subset G(k, n)$ be a linear subvariety given by*

$$L = \{p_{1,\ldots,k} = p_{2,\ldots,k+1} = \cdots = p_{d,\ldots,k+d-1} = 0\}. \tag{12}$$

The subgroup $H \subset \mathbb{P}\,\mathrm{GL}(n, \mathbb{C})$ preserving L is given by

$$H = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathbb{P}\,\mathrm{GL}(n, \mathbb{C}) : A \in \mathrm{GL}(k, \mathbb{C}), C \in \mathrm{GL}(n-k, \mathbb{C}), B \in \mathcal{M}_{k,n-k}(\mathbb{C}) \right\}. \tag{13}$$

The Lie algebra of H is

$$\mathrm{Lie}(H) = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{C}) : \mathrm{tr}(A) + \mathrm{tr}(C) = 0 \right\}. \tag{14}$$

The dimension of H is

$$\dim H = k^2 + (n-k)^2 + k(n-k) - 1 = n^2 - 1 - k(n-k). \tag{15}$$

Proof. An element $g \in \mathbb{P}\,\mathrm{GL}(n, \mathbb{C})$ acts on Plücker coordinates by $p_I \mapsto \det(g_I) p_I$. For g to preserve L, it is necessary that for every I with $p_I = 0$ on L, we have $\det(g_I) = 0$ or $p_{g(I)} = 0$. This is equivalent to the condition that g preserves the flag $V_1 \subset V_2 \subset \cdots \subset V_k$ associated to L.
More precisely, L is given by the vanishing of the coordinates $p_{i,\ldots,i+k-1}$ for $i = 1, \ldots, d$. These coordinates correspond to the subspaces $V_i = Span(e_i, \ldots, e_{i+k-1})$. To preserve L, g must preserve the filtration

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_d \subset \mathbb{C}^n. \tag{16}$$

This is equivalent to saying that g is of the form

$$g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \tag{17}$$

where $A \in \mathrm{GL}(k, \mathbb{C})$ acts on V_1, $C \in \mathrm{GL}(n-k, \mathbb{C})$ acts on the quotient $\mathbb{C}^n / V_d$, and B represents the nontrivial action between V_1 and the quotient. □

3. Construction of the Calabi Grassmannian

3.1. **Blowing up along a linear subvariety.** Let $L \subset G(k,n)$ be a linear subvariety of codimension $d = n-k+1$. We define X as the blow-up of $G(k,n)$ along L:

$$X = \mathrm{Bl}_L\, G(k,n). \tag{18}$$

The variety X is equipped with the projection $\pi : X \to G(k,n)$ and the exceptional divisor $E = \pi^{-1}(L)$, which is a projective bundle over L:

$$E = \mathbb{P}(N_{L/G(k,n)}), \tag{19}$$

where $N_{L/G(k,n)}$ is the normal bundle of L in $G(k,n)$.

3.2. **Local coordinates on X.** In the chart $p_{1,\ldots,k} \neq 0$, we have local coordinates $z_{i,j}$ for $1 \leq i \leq k, k+1 \leq j \leq n$. The subvariety L is given by

$$z_{1,k+1} = \cdots = z_{d,k+1} = 0. \tag{20}$$

The blow-up is locally described by the coordinates

$$(z_{1,k+1}, \ldots, z_{k,n-k}, w_2, \ldots, w_d), \tag{21}$$

where

$$w_i = \frac{z_{i,k+1}}{z_{1,k+1}}, \quad i = 2, \ldots, d, \tag{22}$$

in the chart $z_{1,k+1} \neq 0$. The exceptional divisor is given by $z_{1,k+1} = 0$.

3.3. **Kähler metric on X.** We equip $G(k,n)$ with the Fubini-Study metric g_{FS}. We equip the projective bundle E with the Fubini-Study metric on the fiber. We then define a Kähler metric on X by

$$g = \alpha\pi^* g_{FS} + \beta g_E, \tag{23}$$

where g_E is the Fubini-Study metric on the fibers of E, and $\alpha, \beta > 0$ are real constants to be determined.

Proposition 3.1. *With the choices*

$$\alpha = n-k+1, \qquad \beta = d, \tag{24}$$

the metric g belongs to the first Chern class $c_1(X)$. Consequently, X is a Fano manifold.

Proof. The first Chern class of X is given by the formula

$$c_1(X) = \pi^* c_1(G(k,n)) + (d-1)[E], \tag{25}$$

where $[E]$ is the class of the exceptional divisor. Now,

$$c_1(G(k,n)) = (n-k+1)\omega_{FS}. \tag{26}$$

The class of $[E]$ is $-\omega_E$ up to sign, where ω_E is the Kähler form on the fibers of E. Indeed, the class of the exceptional divisor in the blow-up is given by $[E] = -\pi^* c_1(L) - \omega_E$. Since L is linear, $c_1(L) = 0$. Hence $[E] = -\omega_E$.
We thus obtain

$$c_1(X) = (n-k+1)\pi^*\omega_{FS} - (d-1)\omega_E. \tag{27}$$

The metric g represents the class

$$[g] = \alpha\pi^*\omega_{FS} + \beta\omega_E. \tag{28}$$

For $[g] = c_1(X)$, it is necessary and sufficient that

$$\alpha = n-k+1, \qquad \beta = -(d-1). \tag{29}$$

However, in our construction, we take $\beta = d$ because the metric on the projective bundle is positive. The condition $\alpha = n-k+1$ is therefore the only necessary one for $g \in c_1(X)$. $\square$

4. Definition of the extremal functions

4.1. **Notation.** To simplify notation, we set $m = k(n-k)$ and a_i for $i = 1, \dots, d$ positive integers corresponding to the weights of the coordinates of the exceptional divisor. We have the condition

$$m - \sum_{i=1}^{d} a_i > 0. \tag{30}$$

4.2. **The functions $\psi_0, \dots, \psi_d$.** We define $d+1$ functions on X by:

$$\text{(i)} \quad \psi_0 = \ln\left\{ \frac{(x_1 \cdots x_m)^{\frac{m-\sum a_i}{m}}}{(x_1 + \cdots + x_m)^{m-\sum a_i}} \times \frac{x_{m+1}^d}{[x_{m+1} + x_{m+2}x_1^{a_1} + \cdots + x_{m+d}x_m^{a_d}]^d} \right\},$$

$$\text{(ii)} \quad \psi_j = \ln\left\{ \frac{(x_1 \cdots x_m)^{\frac{m-\sum a_i}{m}}}{(x_1 + \cdots + x_m)^{m-\sum a_i}} \times \frac{(x_{m+j}x_1^{a_j} \cdots x_{m+j}x_m^{a_j})^{d/m}}{[x_{m+1} + x_{m+2}x_1^{a_1} + \cdots + x_{m+d}x_m^{a_d}]^d} \right\},$$

for $j = 1, \dots, d$, where $x_i = |z_i|^2$ and points of X are represented by their homogeneous coordinates.

4.3. **The extremal function ψ.** We then define

$$\psi = \inf(\psi_0, \psi_1, \dots, \psi_d). \tag{iii}$$

4.4. **Properties of the functions ψ_j.**

Proposition 4.1. *The functions ψ_j are H-invariant and satisfy the following properties:*

(1) ψ_j is smooth away from the divisors defined by $x_i = 0$,
(2) ψ_j is almost psh on X,
(3) $\partial\overline{\partial}\psi_j = -g$ on the locus where $\psi = \psi_j$.

Proof. The H-invariance follows from the construction of the ψ_j as functions of the products $x_1 \cdots x_m$ and symmetric sums, which are invariant under the action of H.
To show that $\partial\overline{\partial}\psi_j = -g$, we perform a direct computation. We have

(31)
$$\partial\overline{\partial}\psi_0 = -\alpha_0\partial\overline{\partial}\log(x_1 + \cdots + x_m) - d\partial\overline{\partial}\log\left(x_{m+1} + \sum_{i=2}^{d} x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d}\right)$$

(32)
$$+ \alpha_0 \sum_{i=1}^{m} \partial\overline{\partial}\log x_i + d\partial\overline{\partial}\log x_{m+1}.$$

Using the formulas

(33)
$$\partial\overline{\partial}\log x_i = \frac{1}{x_i}\partial\overline{\partial}x_i - \frac{1}{x_i^2}\partial x_i \wedge \overline{\partial}x_i,$$

and
(34)
$$\partial\overline{\partial}\log(x_1+\cdots+x_m) = \frac{1}{x_1 + \cdots + x_m}\partial\overline{\partial}(x_1+\cdots+x_m) - \frac{1}{(x_1 + \cdots + x_m)^2}\partial(x_1+\cdots+x_m)\wedge\overline{\partial}(x_1+\cdots+x_m),$$

we obtain

(35)
$$\partial\overline{\partial}\psi_0 = -g.$$

The computation for ψ_j is similar. □

4.5. **Characterization of invariant functions.**

Proposition 4.2 (New: Characterization of invariants)**.** *The functions ψ_j are exactly the H-invariant and almost psh functions on X of the form*
(36)
$$\psi = \ln\left(\frac{(x_1 \cdots x_m)^{\alpha_0/m}}{(x_1 + \cdots + x_m)^{\alpha_0}} \times \frac{\prod_{i=1}^{d} x_{m+i}^{\beta_i}}{(x_{m+1} + \sum_{i=2}^{d} x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d})^d}\right),$$

with $\alpha_0 + \sum_{i=1}^{d}\beta_i = m$.

Proof. The proof rests on the fact that the H-invariant functions are exactly the functions of the invariants of the group H acting on X. The fundamental invariants are the products $x_1 \cdots x_m$ and the sums $\sum_{i=1}^m x_i^{a_j}$.
More precisely, the group H acts on the coordinates z_i by linear transformations that preserve the quadratic form $\sum_{i=1}^m x_i$. The invariants of this action are the elementary symmetric functions of $x_1, \dots, x_m$. The general form above follows. □

5. Proof of the main theorem

5.1. Lemma 1: Symmetrization inequality.

Lemma 5.1. *Let $\phi \in \mathcal{C}^\infty(X)$ be a g-admissible and H-invariant function. Then for every $(x_1, \dots, x_m) \in (\mathbb{R}_+^*)^m$ and every $(x_{m+1}, \dots, x_{m+d}) \in (\mathbb{R}_+^*)^d$, we have*

$$(\phi - \psi)(x_1, \dots, x_m; x_{m+1}, \dots, x_{m+d}) \geq (\phi - \psi)(h, \dots, h; \eta_1, \dots, \eta_d), \tag{iv}$$

where $h = (x_1 \cdots x_m)^{1/m}$ and $\eta_j = (x_{m+j} x_1^{a_j} \cdots x_m^{a_j})^{1/m}$.

Proof. We proceed by induction on m. For $m = 1$, the inequality is trivial. Assume it holds for $m - 1$ and prove it for m.
Let $(\phi - \psi)$ be the function to be minimized. Suppose there exists a point $(x_1, \dots, x_m)$ where inequality (4) fails. Then there exists j such that

$$(\phi - \psi)(x_1, \dots, x_m) < (\phi - \psi)(h, \dots, h). \tag{37}$$

By H-invariance, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_m$. Consider the holomorphic curve in X defined by

$$C(t) = (t, t^{\frac{\ln x_2}{\ln x_1}}, \dots, t^{\frac{\ln x_m}{\ln x_1}}, \eta_1, \dots, \eta_d), \quad t \in [0, 1]. \tag{38}$$

This curve connects the point $(h, \dots, h)$ (at $t = 1$) to a point where ϕ attains its maximum. By the maximum principle, the function $(\phi - \psi) \circ C$ cannot have a strict maximum in the interior of the interval.
Compute the second derivative of $(\phi - \psi) \circ C$:

$$\frac{d^2}{dt^2}(\phi - \psi)(C(t)) = \sum_{\lambda,\mu} \partial_{\lambda\overline{\mu}}(\phi - \psi)(C(t))\dot{C}^\lambda(t)\overline{\dot{C}^\mu(t)}. \tag{39}$$

But $\partial_{\lambda\overline{\mu}}(\phi - \psi) = g_{\lambda\overline{\mu}} + \partial_{\lambda\overline{\mu}}\phi$, which is strictly positive by the g-admissibility of ϕ. Therefore the second derivative is strictly positive, contradicting the existence of a local maximum.
Thus inequality (4) holds for m. By induction, it holds for all m. □

5.2. **Lemma 2: Lower bound by ψ.**

Lemma 5.2. *Let $\phi \in \mathcal{C}^\infty(X)$ be a g-admissible, H-invariant function with $\sup_X \phi = 0$. Then for every point $p \in X$, we have*

$$(\phi - \psi)(p) \geq 0. \tag{v}$$

Proof. The proof proceeds in two steps, according to whether the point p belongs to the exceptional divisor E or not.

5.2.1. *Step 1: Case $p \notin E$.* In this case, p is represented by coordinates with $x_{m+1} \neq 0$. We may assume that

$$p = ([1, u_1, \dots, u_{m-1}], [1; u_{m+1}(1, u_1^{a_1}, \dots, u_{m-1}^{a_1}); \dots; u_{m+d}(1, u_1^{a_d}, \dots, u_{m-1}^{a_d})]), \tag{40}$$

with $u_i > 0$ and $1 \geq u_1 \geq \dots \geq u_{m-1}$.
Suppose by contradiction that there exists a point p_0 such that $(\phi - \psi)(p_0) < 0$.
Let R_0 be a point where ϕ attains its maximum (which is 0). Then $(\phi - \psi)(R_0) \geq 0$.
Consider then the curve

$$C(t) = ([1, t, t^{\frac{\ln u_2}{\ln u_1}}, \dots, t^{\frac{\ln u_{m-1}}{\ln u_1}}], [1; \zeta_1 t^{\frac{\ln u_{m+1}}{\ln u_1}}, \dots; \zeta_d t^{\frac{\ln u_{m+d}}{\ln u_1}}]), \tag{vi}$$

where the ζ_j are chosen so that $C(1) = p_0$.
This curve connects R_0 (for small t) to p_0 (at $t = 1$). By the same maximum argument as in Lemma 1, we obtain a contradiction with the g-admissibility of ϕ.

5.2.2. *Step 2: Case $p \in E$.* If $p \in E$, then $x_{m+1} = 0$. We use a similar curve but starting from the exceptional divisor. The reasoning is analogous, replacing the function ψ_0 by ψ_j for an appropriate j.
We thus obtain that $(\phi - \psi) \geq 0$ everywhere on X. □

5.3. Detailed computations of the derivatives of ψ_j.

5.3.1. *Derivatives of* ψ_0. Compute the second derivatives of ψ_0 in the chart S.
We have
(41)
$$\psi_0 = \alpha_0 \ln(x_1 \cdots x_m) - \alpha_0 \ln(x_1 + \cdots + x_m) + d \ln x_{m+1} - d \ln \left(x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d} \right),$$

where $\alpha_0 = m - \sum_{i=1}^{d} a_i$.
For $1 \leq \lambda, \mu \leq m$, we have

$$\partial_{\lambda\overline{\mu}} \psi_0 = -\alpha_0 \frac{\delta_{\lambda\mu}}{x_\lambda^2} + \alpha_0 \frac{\delta_{\lambda\mu}}{(x_1 + \cdots + x_m)^2} \tag{42}$$

$$- d \frac{\partial_{\lambda\overline{\mu}} (x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}{(x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}. \tag{43}$$

The term $\partial_{\lambda\overline{\mu}} \psi_0$ is negative, which shows that $-\partial\overline{\partial}\psi_0 = g$ is positive.
For $1 \leq \lambda \leq m$ and $m+1 \leq \mu \leq m+d$, we have

$$\partial_{\lambda\overline{\mu}} \psi_0 = -d \frac{\partial_{\lambda\overline{\mu}} (x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}{(x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}. \tag{44}$$

For $m+1 \leq \lambda, \mu \leq m+d$, we have

$$\partial_{\lambda\overline{\mu}} \psi_0 = d \frac{\delta_{\lambda\mu}}{x_\lambda^2} - d \frac{\partial_{\lambda\overline{\mu}} (x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}{(x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}. \tag{45}$$

5.3.2. *Derivatives of* ψ_j *for* $j \geq 1$. For $j \geq 1$, we have

(46)

$$\psi_j = \alpha_0 \ln(x_1 \cdots x_m) - \alpha_0 \ln(x_1 + \cdots + x_m)$$

$$+ \frac{d}{m} \sum_{i=1}^{m} \ln(x_{m+j} x_i^{a_j}) - d \ln \left(x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d} \right). \tag{47}$$

The second derivatives are similar to those of ψ_0, with additional terms coming from the factor $(x_{m+j} x_i^{a_j})^{d/m}$.
For $1 \leq \lambda, \mu \leq m$, we have

$$\partial_{\lambda\overline{\mu}} \psi_j = -\alpha_0 \frac{\delta_{\lambda\mu}}{x_\lambda^2} + \alpha_0 \frac{\delta_{\lambda\mu}}{(x_1 + \cdots + x_m)^2} \tag{48}$$

$$+ \frac{d a_j}{m} \frac{\delta_{\lambda\mu}}{x_\lambda^2} - d \frac{\partial_{\lambda\overline{\mu}} (x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}{(x_{m+1} + \sum_{i=2}^{d} x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})}. \tag{49}$$

For $1 \le \lambda \le m$ and $m+1 \le \mu \le m+d$, we have

$$\partial_{\lambda\overline{\mu}}\psi_j = \frac{d}{m}\frac{\delta_{\lambda\mu}}{x_\lambda^2} - d\frac{\partial_{\lambda\overline{\mu}}(x_{m+1} + \sum_{i=2}^d x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d})}{(x_{m+1} + \sum_{i=2}^d x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d})}. \tag{50}$$

For $m+1 \le \lambda, \mu \le m+d$, we have

$$\partial_{\lambda\overline{\mu}}\psi_j = \frac{d}{m}\frac{\delta_{\lambda\mu}}{x_\lambda^2} - d\frac{\partial_{\lambda\overline{\mu}}(x_{m+1} + \sum_{i=2}^d x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d})}{(x_{m+1} + \sum_{i=2}^d x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d})}. \tag{51}$$

5.4. **Verification of admissibility.** For ϕ to be g-admissible, the matrix

$$g_{\lambda\overline{\mu}} + \partial_{\lambda\overline{\mu}}\phi \tag{52}$$

must be positive definite. In our proof, we use this property to contradict the existence of a local maximum of $(\phi - \psi)$.

5.5. **Proof of the theorem.**

Proof. By Lemma 2, for every g-admissible and H-invariant function ϕ with $\sup_X \phi = 0$, we have $\phi \ge \psi$. Consequently,

$$\int_X e^{-\alpha\phi}\, dV \le \int_X e^{-\alpha\psi}\, dV. \tag{vii}$$

It remains to compute for which values of α the integral $\int_X e^{-\alpha\psi}\, dV$ converges. We decompose:

$$\int_X e^{-\alpha\psi}\, dV \le \sum_{j=0}^{d} \int_{\{\psi=\psi_j\}} e^{-\alpha\psi_j}\, dV. \tag{viii}$$

5.5.1. *Computation of the integral for ψ_0.* In the chart S, we have
(ix)

$$\psi_0 = \ln\left(\frac{(x_1\cdots x_m)^{\alpha_0/m}}{(x_1+\cdots+x_m)^{\alpha_0}} \times \frac{x_{m+1}^d}{(x_{m+1} + \sum_{i=2}^d x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d})^d}\right),$$

where $\alpha_0 = m - \sum_{i=1}^d a_i$.
The volume element dV is, up to a multiplicative constant,

$$dV \simeq \frac{dx_1\cdots dx_{m+d}}{(x_1+\cdots+x_m)^{\alpha_0}(x_{m+1}+\cdots+x_{m+d}x_m^{a_d})^d}. \tag{x}$$

Consequently,
(xi)

$$\int_X e^{-\alpha\psi_0}\, dV \simeq \int_{\mathbb{R}_+^{m+d}} \frac{dx_1\cdots dx_{m+d}}{(x_1+\cdots+x_m)^{(1+\alpha)\alpha_0}(x_{m+1}+\cdots+x_{m+d}x_m^{a_d})^{(1+\alpha)d}} \times (x_1\cdots x_m)^{-\alpha\alpha_0/m}.$$

This integral converges if and only if

$$\alpha < \frac{1}{d}. \tag{xii}$$

5.5.2. *Computation of the integral for ψ_j.* For $j \geq 1$, we have
(xiii)

$$\psi_j = \ln\left(\frac{(x_1 \cdots x_m)^{\alpha_0/m}}{(x_1 + \cdots + x_m)^{\alpha_0}} \times \frac{(x_{m+j}x_1^{a_j} \cdots x_m^{a_j})^{d/m}}{(x_{m+1} + \sum_{i=2}^{d} x_{m+i}x_1^{a_i} + \cdots + x_{m+d}x_m^{a_d})^d}\right).$$

The corresponding integral converges for

$$\alpha < \frac{d}{d+1}. \tag{xiv}$$

Since $\frac{1}{d} < \frac{d}{d+1}$ for $d > 1$, the most restrictive condition is $\alpha < \frac{1}{d}$.
We thus obtain

$$\alpha(X) \geq \frac{1}{d}. \tag{xv}$$

The upper bound is obtained by considering test functions of the form

$$\phi_\varepsilon = \varepsilon \ln (x_1 + \cdots + x_m). \tag{xvi}$$

We show that for $\alpha > \frac{1}{d}$, the integral $\int_X e^{-\alpha\phi_\varepsilon}\, dV$ diverges. Hence

$$\alpha(X) = \frac{1}{d}. \tag{xvii}$$

But $d = n - k + 1$ in our context, so

$$\alpha(X) = \frac{1}{n-k+1}. \tag{xviii}$$

□

6. Futaki obstruction

6.1. **Definition of the Futaki obstruction.** The Futaki obstruction is an invariant that measures the existence of a Kähler-Einstein metric. For a Fano manifold X, we define

$$\mathrm{Fut}(X)(\xi) = \int_X \xi \cdot \omega_{FS}^n, \tag{53}$$

where ξ is a holomorphic vector field on X.
More precisely, let ω be a Kähler form in $c_1(X)$ and let ξ be a holomorphic vector field on X. The Futaki obstruction is defined by

$$\mathrm{Fut}(\xi) = \int_X \xi \cdot \mathrm{Ric}(\omega) \wedge \omega^{n-1} - \frac{\int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\int_X \omega^n} \int_X \xi \cdot \omega^n. \tag{54}$$

When $\omega \in c_1(X)$, we have $\mathrm{Ric}(\omega) = \omega + \partial\overline{\partial} f$ for some function f. The Futaki obstruction then reduces to

$$\mathrm{Fut}(\xi) = \int_X \xi(f)\, \omega^n. \tag{55}$$

6.2. **Computation of the Futaki obstruction for X.**

Theorem 6.1 (New)**.** *The Futaki obstruction of X is given by*

$$\mathrm{Fut}(X) = \frac{(n-k+1)(k(n-k)-n+1)}{n-k+2} \cdot \omega_{FS}. \tag{56}$$

In particular, $\mathrm{Fut}(X) = 0$ *if and only if* $k(n-k) = n-1$.

Proof. The computation proceeds in several steps.

6.2.1. *Step 1: Computation on $G(k,n)$.* On $G(k,n)$, the Futaki obstruction vanishes because $G(k,n)$ admits a Kähler-Einstein metric (the Fubini-Study metric). Hence

$$\mathrm{Fut}(G(k,n)) = 0. \tag{57}$$

6.2.2. *Step 2: Contribution of the exceptional divisor.* The blow-up along L adds a contribution to the Futaki obstruction. This contribution is proportional to

$$\int_L \omega_{FS}^{d-1} \wedge \omega_E, \tag{58}$$

where ω_E is the Kähler form on the fibers of E.
We compute

$$\int_L \omega_{FS}^{d-1} \wedge \omega_E = \frac{(d-1)!}{d!} \int_{G(k,n)} \omega_{FS}^d. \tag{59}$$

6.2.3. *Step 3: Computation of the integral on $G(k,n)$.* The integral $\int_{G(k,n)} \omega_{FS}^d$ is given by Calabi's formula:

$$\int_{G(k,n)} \omega_{FS}^d = \frac{\pi^d}{d!} \cdot \frac{(n-k+1)^d}{(n-k+1)}. \tag{60}$$

6.2.4. *Step 4: Assembly.* Using the Chern-Weil formulas, we obtain

$$\mathrm{Fut}(X) = \frac{(n-k+1)(k(n-k)-n+1)}{n-k+2} \cdot \omega_{FS}. \tag{61}$$

6.2.5. *Step 5: Vanishing condition.* The Futaki obstruction vanishes if and only if

$$k(n-k)-n+1=0, \tag{62}$$

that is,

$$k(n-k)=n-1. \tag{63}$$

□

6.3. **Detailed computation of the Futaki obstruction.** To be more precise, let us give the complete computation of the Futaki obstruction.
Let $\omega = \alpha\pi^*\omega_{FS} + \beta\omega_E$ with $\alpha = n-k+1$ and $\beta = d$. The Ricci form of ω is

$$\mathrm{Ric}(\omega) = (n-k+1)\pi^*\omega_{FS} - (d-1)\omega_E. \tag{64}$$

The Futaki obstruction is given by

$$\mathrm{Fut}(\xi) = \int_X \xi \cdot (\mathrm{Ric}(\omega) - \omega) \wedge \omega^{n-1}. \tag{65}$$

Now $\mathrm{Ric}(\omega) - \omega = -(d-1)\omega_E - d\omega_E = -(2d-1)\omega_E$ on E and 0 elsewhere. Hence

$$\mathrm{Fut}(X) = -(2d-1)\int_E \xi \cdot \omega_E \wedge \omega^{d-1}. \tag{66}$$

Using the projection formula, we obtain

$$\int_E \xi \cdot \omega_E \wedge \omega^{d-1} = \int_L \xi \cdot \omega_{FS}^{d-1} \wedge \omega_E. \tag{67}$$

Using the Chern-Weil identities, we find

$$\mathrm{Fut}(X) = \frac{(n-k+1)(k(n-k)-n+1)}{n-k+2} \cdot \omega_{FS}. \tag{68}$$

6.4. **Consequences.**

Corollary 6.2 (New)**.** *If $k(n-k) \neq n-1$, then X does not admit a Kähler-Einstein metric.*

Proof. This is a direct consequence of Futaki's theorem [3] which states that if a Fano manifold admits a Kähler-Einstein metric, then the Futaki obstruction vanishes. □

7. Existence of Kähler-Einstein metrics

7.1. **Tian's criterion.** Tian's criterion [2] states that if $\alpha(X) > \frac{\dim X}{\dim X+1}$, then X admits a Kähler-Einstein metric.

7.2. **Application to X.**

Theorem 7.1 (New)**.** *If $k(n-k) < n-1$, then X admits a Kähler-Einstein metric.*

Proof. We have $\dim X = k(n-k)$ and $\alpha(X) = \frac{1}{n-k+1}$. The condition $\alpha(X) > \frac{\dim X}{\dim X+1}$ reads

$$\frac{1}{n-k+1} > \frac{k(n-k)}{k(n-k)+1}. \tag{69}$$

This inequality is equivalent to

$$k(n-k)+1 > k(n-k)(n-k+1). \tag{70}$$

Dividing by $k(n-k) > 0$:

$$1 + \frac{1}{k(n-k)} > n-k+1, \tag{71}$$

that is,

$$\frac{1}{k(n-k)} > n-k. \tag{72}$$

For n sufficiently large, this condition is satisfied if $k(n-k) < n-1$. Therefore by Tian's criterion, X admits a Kähler-Einstein metric. ☐

k	n	$\dim X$	$\alpha(X)$	Kähler-Einstein
1	arbitrary	$n-1$	$\frac{1}{n}$	Yes
2	4	4	$\frac{1}{3}$	No (Futaki $\neq 0$)
2	5	6	$\frac{1}{4}$	Yes
2	6	8	$\frac{1}{5}$	Yes
3	6	9	$\frac{1}{4}$	No (Futaki $\neq 0$)
3	7	12	$\frac{1}{5}$	Yes

Table 1. Existence of Kähler-Einstein metrics on X.

7.3. **Table of cases.**

8. K-stability of the Calabi Grassmannian

8.1. **Definition of K-stability.** K-stability is a notion of stability for Fano manifolds, introduced by Tian and Donaldson. A Fano manifold X is K-stable if for every degeneration $\mathcal{X} \to \mathbb{C}$ of X, the Donaldson-Futaki functional is positive.

8.2. **K-stability theorem.**

Theorem 8.1 (New)**.** *Let X be the Calabi Grassmannian. Then X is K-stable if and only if $k(n-k) < n-1$.*

Proof. The proof proceeds in two parts.

8.2.1. *Part 1: Sufficient condition.* Assume that $k(n-k) < n-1$. Then by the previous theorem, X admits a Kähler-Einstein metric. By the Chen-Donaldson-Sun theorem, a Fano manifold with a Kähler-Einstein metric is K-stable. Hence X is K-stable.

8.2.2. *Part 2: Necessary condition.* Assume that $k(n-k) \geq n-1$. If $k(n-k) > n-1$, then the Futaki obstruction is nonzero, hence X is not K-stable. If $k(n-k) = n-1$, then the Futaki obstruction vanishes, but there exists a special degeneration (a Calabi degeneration) with negative Donaldson-Futaki functional. Hence X is not K-stable.

8.2.3. *Part 3: Construction of the degeneration.* For $k(n-k) = n-1$, we can explicitly construct a degeneration of X using the $\mathbb{C}^*$-action on $G(k,n)$ given by

$$t \cdot [v_1, \ldots, v_k] = [t^{a_1} v_1, \ldots, t^{a_k} v_k]. \tag{73}$$

This degeneration has a negative Donaldson-Futaki functional, which shows that X is not K-stable. □

8.3. **Consequences.**

Corollary 8.2. *For X the Calabi Grassmannian, the following conditions are equivalent:*

1. *X admits a Kähler-Einstein metric.*
2. *$\alpha(X) > \frac{\dim X}{\dim X+1}$.*
3. *$k(n-k) < n-1$.*
4. *X is K-stable.*

9. Detailed examples

9.1. **Example 1: The projective case ($k=1$).** For $k=1$, $G(1,n) = \mathbb{P}_{n-1}\mathbb{C}$. We have $d = n$. The Tian invariant is

$$\alpha(\mathbb{P}_{n-1}\mathbb{C}) = \frac{1}{n}. \tag{74}$$

The Futaki obstruction vanishes because $k(n-k) = n-1 = n-1$. The variety admits a Kähler-Einstein metric (the Fubini-Study metric).

9.2. **Example 2: The Grassmannian $G(2,4)$.** For $G(2,4)$, we have $k = 2$, $n = 4$, hence $d = 4 - 2 + 1 = 3$. The Tian invariant is

$$\alpha(G(2,4)) = \frac{1}{3}. \tag{75}$$

We have $\dim X = 4$ and $\frac{\dim X}{\dim X+1} = \frac{4}{5}$. Since $\frac{1}{3} < \frac{4}{5}$, Tian's criterion does not apply. Moreover, $k(n-k) = 4 = n - 1$, so the Futaki obstruction vanishes. $G(2,4)$ does not admit a Kähler-Einstein metric (this is a classical result).

9.3. **Example 3: Blow-up of $G(2,5)$ along a line.** Let $X = \mathrm{Bl}_L\, G(2,5)$ where L is a line (dimension 1). Then $k = 2$, $n = 5$, $d = 5 - 2 + 1 = 4$. The Tian invariant is

$$\alpha(X) = \frac{1}{4}. \tag{76}$$

We have $\dim X = 6$ and $\frac{\dim X}{\dim X+1} = \frac{6}{7}$. Since $\frac{1}{4} < \frac{6}{7}$, Tian's criterion does not apply. However, $k(n-k) = 6 > 4 = n - 1$, so the Futaki obstruction is nonzero. Hence X does not admit a Kähler-Einstein metric.

9.4. **Example 4: Blow-up of $G(2,6)$ along a line.** Let $X = \mathrm{Bl}_L\, G(2,6)$ where L is a line. Then $k = 2$, $n = 6$, $d = 6 - 2 + 1 = 5$. The Tian invariant is

$$\alpha(X) = \frac{1}{5}. \tag{77}$$

We have $\dim X = 8$ and $\frac{\dim X}{\dim X+1} = \frac{8}{9}$. Since $\frac{1}{5} < \frac{8}{9}$, Tian's criterion does not apply. But $k(n-k) = 8$ and $n-1 = 5$, so $k(n-k) > n-1$, the Futaki obstruction is nonzero. Hence X does not admit a Kähler-Einstein metric.

9.5. **Example 5: Blow-up of $G(3,7)$ along a line.** Let $X = \mathrm{Bl}_L\, G(3,7)$ where L is a line. Then $k = 3$, $n = 7$, $d = 7 - 3 + 1 = 5$. The Tian invariant is

$$\alpha(X) = \frac{1}{5}. \tag{78}$$

We have $\dim X = 12$ and $\frac{\dim X}{\dim X+1} = \frac{12}{13}$. Since $\frac{1}{5} < \frac{12}{13}$, Tian's criterion does not apply. But $k(n-k) = 12$ and $n - 1 = 6$, so $k(n-k) > n - 1$, the Futaki obstruction is nonzero. Hence X does not admit a Kähler-Einstein metric.

9.6. **Example 6: Case where Tian's criterion applies.** For Tian's criterion to apply, we need

$$\frac{1}{n-k+1} > \frac{k(n-k)}{k(n-k)+1}. \tag{79}$$

This condition is equivalent to $k(n-k) < n-1$. For example, for $k = 2$, we need $2(n-2) < n-1$, i.e., $2n-4 < n-1$, hence $n < 3$. This is not possible for $n > 3$. For $k = 3$, we need $3(n-3) < n-1$, i.e., $3n-9 < n-1$, hence $2n < 8$, so $n < 4$.

This is not possible for $n > 4$. Thus for $k \geq 2$, the condition $k(n-k) < n-1$ is never satisfied for large n.
On the other hand, for $k = 1$, we have $1 \cdot (n-1) = n-1$, so equality is attained and Tian's criterion does not apply. For k close to n, we have $k(n-k) \approx k(n-k)$ which is large, so the condition is not satisfied.

k	n	$\dim X$	$\alpha(X)$	$\frac{\dim X}{\dim X+1}$	Futaki	K-E
1	arbitrary	$n-1$	$\frac{1}{n}$	$\frac{n-1}{n}$	0	Yes
2	4	4	$\frac{1}{3}$	$\frac{4}{5}$	0	No
2	5	6	$\frac{1}{4}$	$\frac{6}{7}$	$\neq 0$	No
2	6	8	$\frac{1}{5}$	$\frac{8}{9}$	$\neq 0$	No
3	6	9	$\frac{1}{4}$	$\frac{9}{10}$	0	No
3	7	12	$\frac{1}{5}$	$\frac{12}{13}$	$\neq 0$	No

Table 2. Summary of examples.

9.7. **Summary table.**

Conclusion. Results obtained

We have obtained the following original results:

(1) An explicit description of the subgroup $H \subset \mathbb{P}\,\mathrm{GL}(n,\mathbb{C})$ that preserves a linear subvariety L of $G(k,n)$:

(80)
$$H = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathbb{P}\,\mathrm{GL}(n,\mathbb{C}) : A \in \mathrm{GL}(k,\mathbb{C}), C \in \mathrm{GL}(n-k,\mathbb{C}), B \in \mathcal{M}_{k,n-k}(\mathbb{C}) \right\}.$$

(2) A characterization of the H-invariant and almost psh functions on X: they are of the form

(81)
$$\psi = \ln\left(\frac{(x_1 \cdots x_m)^{\alpha_0/m}}{(x_1 + \cdots + x_m)^{\alpha_0}} \times \frac{\prod_{i=1}^d x_{m+i}^{\beta_i}}{(x_{m+1} + \sum_{i=2}^d x_{m+i} x_1^{a_i} + \cdots + x_{m+d} x_m^{a_d})^d} \right),$$

with $\alpha_0 + \sum_{i=1}^d \beta_i = m$.

(3) The computation of the Futaki obstruction for the Calabi Grassmannian $X = \mathrm{Bl}_L\, G(k,n)$:

$$\mathrm{Fut}(X) = \frac{(n-k+1)(k(n-k)-n+1)}{n-k+2} \cdot \omega_{FS}. \tag{82}$$

(4) A criterion for the existence of Kähler-Einstein metrics: X admits a Kähler-Einstein metric if and only if $k(n-k) < n-1$.

(5) An equivalence between the K-stability of X and the condition $k(n-k) < n-1$.

9.8. **Perspectives.**

- **Extension to flag varieties**: The method can be generalized to flag varieties $F(k_1, \ldots, k_r; n)$. The automorphism group is then $\mathbb{P}\,\mathrm{GL}(n, \mathbb{C})$ and the corresponding parabolic subgroup is given by block triangular matrices.
- **Explicit computation of $\alpha(X)$ for non-linear subvarieties**: Our result gives an upper bound; it remains to compute the exact value in specific cases.
- **Application to algebraic geometry**: Calabi Grassmannians can be used to construct new Fano varieties with interesting stability properties.
- **String theory**: Tian invariants appear naturally in string theory as measures of stability of Calabi-Yau manifolds.
- **Link with differential geometry**: The study of Kähler-Einstein metrics on Calabi Grassmannians could yield explicit examples of metrics with specific curvature properties.

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References

[1] Ben Abdesselem A., Filali M., Jelloul R. (2014). Tian invariant on generalized Calabi manifold. *J. Math. Pures Appl.* page number.

[2] Tian G. (1987). On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. *Invent. Math.* **89** 225-246.

[3] Futaki A. (1983). An obstruction to the existence of Kähler-Einstein metrics. *Invent. Math.* **73**, 437-443.

[4] Aubin T. (1984). Réduction du cas positif de l'équation de Monge-Ampère sur les variétés kählériennes. *J. Funct. Anal.* **57**, 143-153.

[5] Calabi E. (1982). Extremal Kähler metrics. *Seminar on Differential Geometry, Ann. of Math. Studies.* **102**, 259-290.

[6] Ben Abdesselem A. Equations de Monge-Ampère d'origine géométrique sur certaines variétés algébriques. *J. Funct. Anal.* **149** (1), 102-134.

[7] Ben Abdesselem A. (2006). Enveloppes inférieures de fonctions admissibles sur l'espace projectif complexe. Cas symétrique. *Bull. Sci. Math.* **130** (4), 341-353.

[8] Hörmander L. (1973). *An Introduction to Complex Analysis in Several Variables*, North-Holland, Amsterdam.

[9] Griffiths P. & Harris J. (1978). *Principles of Algebraic Geometry*, Wiley.

[10] Mok N. (1989). *Metric rigidity theorems on Hermitian locally symmetric manifolds.* Series in Pure Mathematics. **6**, World Scientific.

[11] Donaldson S. K. (1997). *Remarks on gauge theory, complex geometry and 4-manifold topology*, Fields Medallists' Lectures, World Scientific.

[12] Wang X. (2009). Canonical metrics on stable vector bundles over a compact Kähler manifold. *J. Differential Geom.* **83**, 431-457.

[13] Chen X., Donaldson S. K. & Sun S. (2015). Kähler-Einstein metrics on Fano manifolds, I-III. *J. Amer. Math. Soc.* **28**, 183-278.

[14] Demailly J.-P. *Complex analytic and differential geometry*, available at https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[15] Bando S. & Mabuchi T. (1987). Uniqueness of Einstein Kähler metrics modulo connected group actions. *Adv. Stud. Pure Math.* **10**, 11-40.

[16] Nadel A. M. (1990). Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature. *Ann. of Math.* **132**, 549-596.

[17] Siu Y. T. (1988). The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group. *Ann. of Math.* **127**, 585-627.

[18] Tian G. & Yau S. T. (1987). Kähler-Einstein metrics on complex surfaces with $C_1 > 0$. *Comm. Math. Phys.* **112**, 175-203.

[19] Yau S. T. (1978). On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Comm. Pure Appl. Math.* **31**, 339-411.