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Radial Solution of the Monge-Ampère Equation on $\mathbb{R}^3$ and $\mathbb{R}^4$

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Abstract

In this paper, we determine radial solutions of the Monge-Ampère equation on $\mathbb{R}^3$ and $\mathbb{R}^4$ using a symplectic change of variables. This transformation simplifies the formulation of the equation and allows it to be expressed in radial form, thereby reducing the problem to an associated ordinary differential equation. The existence and uniqueness of the solutions are established and explicit expressions are provided.

1. Introduction and Preliminaries

In dimension n, a Monge-Ampère equation is a linear combination of the minors of the Hessian matrix [2]

$$\mathrm{Hess}(f) = \Big(\frac{\partial^2 f}{\partial q_j \partial q_k}\Big)_{j,k=1\ldots n}.$$

Let $\Omega = \dfrac{i}{2}\big(dz_1 \wedge d\overline{z_1} + \ldots + dz_n \wedge d\overline{z_n}\big)$ be the canonical Kähler form on $\mathbb{C}^n$ and $\alpha = dz_1 \wedge \ldots \wedge dz_n$ be the complex volume form. A special Lagrangian submanifold is a real n-submanifold L which is Lagrangian with respect to Ω and which satisfies the special condition

$$\mathrm{Im}(\alpha)|_L = 0.$$

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These submanifolds, introduced by Harvey and Lawson in their famous article *Calibrated Geometries* [5], are minimal submanifolds of $\mathbb{C}^n$, and more generally of Calabi-Yau manifolds, transverse in some sense to complex submanifolds. They have been extensively studied after the construction proposed by Strominger, Yau and Zaslow [7] of mirror partners of Calabi-Yau manifolds based upon a hypothetical special Lagrangian fibration.

Some examples have been given by many authors. We can cite for example Harvey and Lawson [5], Joyce [6] and Bryant [4].

For every smooth function f on $\mathbb{R}^n$, the graph

$$L_f = \left\{ (q + i\frac{\partial f}{\partial q}),\ q \in \mathbb{R}^n \right\}$$

is a Lagrangian submanifold of $\mathbb{C}^n$. The special Lagrangian condition becomes then a differential equation on f, called the special Lagrangian equation: Noting that the Hessian matrix is symmetric, this is due to the formula [2]:

$$\mathrm{Im}(\det(Id + iA)) = \sum_{k=0}^{n} (-1)^k \sigma_{2k+1}(A) = 0$$

where A is a symmetric matrix, i.e.

$$\sum_{k=0}^{n} (-1)^k \sigma_{2k+1}(\mathrm{Hess}(f)) = 0.$$

For $n = 3$ (see [1]):

$$\Delta f - \mathrm{hess}(f) = 0.$$

A symplectic Monge-Ampère equation in dimension 3 associated with a non-degenerate Monge-Ampère structure (Ω, ω) can be brought back by a symplectic change of variables to one of the following equations

$$\begin{cases} \mathrm{hess}(f) = 1 \\ \Delta f - \mathrm{hess}(f) = 0 \\ \Box f + \mathrm{hess}(f) = 0 \end{cases}$$

and it is integrable if and only if

$$\begin{cases} d\left(\frac{\omega}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0 \\ q \text{ is flat} \end{cases}$$

in dimension 3:

$$\Delta f - \mathrm{hess}(f) = 0 \leftrightarrow \omega = dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 - dp_1 \wedge dp_2 \wedge dp_3$$

and for the Hessian equation:

$$\mathrm{hess}(f) = 1 \leftrightarrow \omega = dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3.$$

The geometric structure on $T^*\mathbb{R}^3$ associated with the Lagrangian special equation

$$\Delta f - \mathrm{hess}(f) = 0$$

is the classic Calabi-Yau structure (q, I, Ω, α) with:

$$\begin{cases} q = -\sum\limits_{j=1}^{3} dx_j.dx_j + dy_j.dy_j \\ I = \sum\limits_{j=1}^{3} \left(\frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j\right) \\ \Omega = \sum\limits_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \end{cases}$$

In the same way, the geometric structure associated with the pseudo special Lagrangian equation

$$\Box f + \mathrm{hess}(f) = 0$$

is the Calabi-Yau pseudo structure $(q_-, I_-, \Omega, \alpha)$ with:

$$\begin{cases} q_- = dx_1.dx_1 - dx_2.dx_2 + dx_3.dx_3 + dy_1.dy_1 - dy_2.dy_2 + dx_3.dx_3 \\ I_- = -\frac{\partial}{\partial y_1} \otimes dx_1 + \frac{\partial}{\partial x_1} \otimes dy_1 + \frac{\partial}{\partial y_2} \otimes dx_2 - \frac{\partial}{\partial x_2} \otimes dy_2 - \frac{\partial}{\partial y_3} \otimes dx_3 + \frac{\partial}{\partial x_3} \otimes dy_3 \\ \Omega = \sum\limits_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \end{cases}$$

These two structures are the only associated examples to the elliptic Monge-Ampère equations with constant coefficients; there are only two equations on $\mathbb{R}^6$.

Note that in any almost Calabi-Yau structure (q, I, Ω, α) is associated with the elliptical structure $(\Omega, \mathrm{Re}(\alpha))$. There is therefore a correspondence between almost (pseudo) Calabi-Yau structures and conformal classes of elliptical Monge-Ampère structures.

Concerning the hyperbolic structure this concerns the equation $\mathrm{hess}(f) = 1$ we will have the structure $(q, S, \Omega, \alpha, \beta)$ with:

$$\begin{cases} q = \sum\limits_{j=1}^{3} dx_j.dy_j \\ S = \sum\limits_{j=1}^{3} \left(\frac{\partial}{\partial x_j} \otimes dx_j - \frac{\partial}{\partial y_j} \otimes dy_j\right) \\ \Omega = \sum\limits_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dx_1 \wedge dx_2 \wedge dx_3, \ \beta = dy_1 \wedge dy_2 \wedge dy_3 \end{cases}$$

2. Solution of the first equation $\mathrm{hess}(f) = 1$

Example 1. Radial solution of $\mathrm{hess}(f) = 1$ in $\mathbb{R}^3$. The equation of the determinant of the Hessian is written as follows:

$$\mathrm{hess}(f) = 1.$$

We will subsequently show the radial differential equation, in order to affirm the existence and uniqueness of the solution.

$$u_{x_i} = u' \frac{x_i}{r}$$

$$u_{x_i,x_j} = u'' \frac{x_i x_j}{r^2} + u' \Big(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \Big) = \delta_{ij} \frac{u'}{r} + \Big(\frac{u''}{r^2} - \frac{u'}{r^3} \Big) x_i x_j$$

$$\mathrm{hess}(u) = \Big(\frac{u'}{r}\Big)^3 \det\Big[\delta_{ij} + \frac{r}{u'}\Big(u'' - \frac{u'}{r}\Big)\frac{x_i x_j}{r^2}\Big] = \Big(\frac{u'}{r}\Big)^3 \Big[1 + \frac{r}{u'}\Big(u'' - \frac{u'}{r}\Big)\Big] = \Big(\frac{u'}{r}\Big)^2 u''$$

Note that: $\nabla_X \partial_r = \frac{1}{r}$ if X is tangent to S^2 and $\nabla_X \partial_r = 0$ if $X = \partial_r$, where ∇_X is the Levi-Civita connection.

$$\begin{cases} \mathrm{Hess}(u)(\partial_r, \partial_r) = u'' \\ \mathrm{Hess}(u)(\partial_{\theta_i}, \partial_r) = 0 \\ \mathrm{Hess}(u)(\partial_{\theta_i}, \partial_{\theta_j}) = r u' \delta_{ij} \end{cases}$$

The metric g is written as follows:

$$g = dr^2 + r^2(d\theta^2 + d\varphi^2).$$

With this information we deduce the radial equation of the determinant of the Hessian:

$$\frac{\mathrm{hess}(u)}{\det(g)} = \Big(\frac{u'}{r}\Big)^2 u''.$$

After normalization we can take $\det(g) = 1$, we end up with the equation

$$\Big(\frac{u'}{r}\Big)^2 u'' = 1.$$

We put $v = u'$, we will have:

$$\Big(\frac{v}{r}\Big)^2 v' = 1$$

which gives:

$$\begin{aligned} v^2 v' &= r^2 \\ \frac{v^3}{3} &= \frac{r^3}{3} + C_1 \\ v^3 &= r^3 + C_2 \\ v &= (r^3 + C_2)^{1/3}. \end{aligned}$$

Hence

$$u = \int (r^3 + C_2)^{1/3} \, dr.$$

3. Solution of the second equation $\Delta f - \text{hess}(f) = 0$

Example 2. Radial solution of $\Delta f - \text{hess}(f) = 0$ in $\mathbb{R}^3$. $\Delta f - \text{hess}(f) = 0 \iff \Delta f = \text{hess}(f)$. Let us look for a radial solution of this equation. We write $f(x_1, x_2, x_3) = u(r)$ with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. According to the first paragraph we have $\text{hess}(u) = \left(\frac{u'}{r}\right)^2 u''$ and for Δf:
we set $f(x) = u(r)$, with $r = \|x\|$. We have: $\frac{\partial f}{\partial x_i} = \frac{x_i}{r} u'$ and therefore $\frac{\partial^2 f}{\partial x_i^2} = \frac{x_i^2}{r^2} u'' + \frac{r^2 - x_i^2}{r^3} u'$ then,

$$\Delta f = \sum_{k=1}^{n} \Big(\frac{x_i^2}{r^2} u'' + \frac{r^2 - x_i^2}{r^3} u' \Big) = u'' + \frac{n-1}{r} u'.$$

For $n = 3$, we have $\Delta f = u'' + \frac{2}{r} u'$.
The equation $\Delta f = \text{hess}(f)$ becomes:

$$u'' + \frac{2}{r} u' = \Big(\frac{u'}{r}\Big)^2 u''.$$

We note $v = u'$; we will have:

$$v' + \frac{2}{r} v = \Big(\frac{v}{r}\Big)^2 v'.$$

Multiplying by r^2:

$$r^2 v' + 2rv = v^2 v'.$$

This can be rewritten as:

$$(r^2 v)' = \Big(\frac{v^3}{3}\Big)'.$$

Integrating:

$$r^2 v = \frac{v^3}{3} + C.$$

Taking $v(0) = 0$, we obtain $r^2 v = \frac{v^3}{3}$, so $v^2 = 3r^2$, i.e. $v = \pm\sqrt{3}\, r$. Thus

$$u(r) = \pm \int \sqrt{3}\, r\, dr = \pm \frac{\sqrt{3}}{2} r^2 + k$$

where k is a real constant.

4. Radial solution of the Monge-Ampère equation on ^{4}

Example 3. Radial solution of $\text{hess}(f) = \pm 1$ **in $\mathbb{R}^4$.** The Monge-Ampère operators theory developed by Lychagin generalizes this correspondence between "calibrated" Lagrangian submanifolds of $\mathbb{R}^{2n}$ and Monge-Ampère equations on $\mathbb{R}^n$.
For every holomorphic function $\phi = f + ig : \mathbb{C}^2 \to \mathbb{C}$, we have (see [1] and [2])

$$\text{hess}_{\mathbb{R}} f = |\, \text{hess}_{\mathbb{C}} \phi|^2.$$

Thus the complex reduction of $\operatorname{hess}_{\mathbb{R}} f = \pm 1$ are

$$|\operatorname{hess}_{\mathbb{C}} \phi|^2 = \pm 1.$$

The equation of the determinant of the Hessian $\operatorname{hess}(f) = 1$ or -1 is equivalent to the equation of the radial function $u(r)$ with $\operatorname{hess}(u) = 1$ or -1 because $f(x) = u(r)$ with $r = \|x\|$.
We have $u_{x_i} = u' \frac{x_i}{r}$ and

$$u_{x_i x_j} = u'' \frac{x_i x_j}{r^2} + u' \Big(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \Big) = \frac{u'}{r} \delta_{ij} + \Big(\frac{u''}{r^2} - \frac{u'}{r^3} \Big) x_i x_j,$$

which implies:

$$\operatorname{hess}(u) = \Big(\frac{u'}{r} \Big)^4 \det \Big[\delta_{ij} + \frac{r}{u'} \Big(u'' - \frac{u'}{r} \Big) \frac{x_i x_j}{r^2} \Big].$$

Hence:

$$\operatorname{hess}(u) = \Big(\frac{u'}{r} \Big)^4 \Big[1 + \frac{r}{u'} \Big(u'' - \frac{u'}{r} \Big) \Big] = \Big(\frac{u'}{r} \Big)^3 u''.$$

Using polar coordinates $(r, \theta_1, \theta_2, \theta_3)$, the Riemannian metric $g = dr^2 + r^2 \sum_{i=1}^3 d\theta_i^2$ and the Levi-Civita connection $\nabla_X \partial_r = \frac{1}{r}$ if X is tangent to S^3 and $\nabla_X \partial_r = 0$ if $X = \partial_r$ we have:

$$\begin{cases} \operatorname{Hess}(u)(\partial_r, \partial_r) = u'' \\ \operatorname{Hess}(u)(\partial_{\theta_i}, \partial_r) = 0 \\ \operatorname{Hess}(u)(\partial_{\theta_i}, \partial_{\theta_j}) = r u' \delta_{ij} \end{cases}$$

We will then have:

$$\frac{\operatorname{hess}(u)}{\det(g)} = \Big(\frac{u'}{r} \Big)^3 u''.$$

Taking $\det(g) = 1$:

$$(u')^3 u'' = r^3.$$

We set $v = u'$; then $v^3 v' = r^3$, i.e. $\frac{v^4}{4} = \frac{r^4}{4} + C$, or equivalently $v^4 = r^4 + C'$. Thus $v = (r^4 + C')^{1/4}$ and therefore:

$$u = \int (r^4 + C')^{1/4} \, dr.$$

Likewise for $\operatorname{hess}(f) = -1$ we will have:

$$u = - \int (r^4 + C')^{1/4} \, dr.$$

The constant C' is determined by the boundary conditions, which proves the existence and uniqueness of the solution.

Conclusion. The search for radial solutions in the field of partial differential equations is an effective tool for showing the existence, uniqueness and explicit expression of the solution and allows us to simplify complex equations. In perspective, we will try to generalize this result to any dimension, namely on ^{n} for $n > 4$.

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