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Polynomial Bounds for Bi-Univalent Functions Associated with Sigmoid Function and $Q-$ Derivative Over a Chebyshev Polynomials

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Abstract

This paper introduces a new subclass of the function class $\sum$ of bi-univalent analytic functions defined in the open unit disk, associated with a certain operator and the Sigmoid function via Chebyshev polynomials. Bounds for the initial coefficients are obtained for the defined class, along with results for the Fekete–Szegö functional. Furthermore, by varying the parameters involved, several results are derived which extend existing findings in the literature.

1. Introduction and Preliminaries

Let A denotes the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized with $f(0) = f'(0) - 1 = 0$ in U. Also let $S \in A$ be the class of analytic univalent in the unit disk U.

A function $f \in A$ is bi-univalent in U if f^{-1} exist and it is univalent in U. The

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concept of bi-univalent was introduced in [20] and the first coefficient bounds was estimated. Other researchers, the likes of [19], [32], [14], [23], [24], [31] further investigation on bi-univalency and they obtained very useful results,(see also [5], [6], [7], [8], [16], [25] & [33]) and the literature therein. furthermore, we note that for $f \in A$ there exist f^{-1} (inverse) satisfying

$$f^{-1}(f(z)) = z, \quad (z \in A)$$

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$(2)\quad g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4.$$

Here we denote by $\sum$ the class of bi-univalent function.
The study of Chebyshev polynomial in the space of univalent function theory has gathered momentum in the recent time due to its both experimental and practical application. This is evident in the literature [1], [2], [11], [16] & [30] and the the literature therein. The classical Chebyshev polynomials of the first and second kinds were first studied by Chebyshev. The first kind of Chebyshev polynomials is defined as follows

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t)$$

From the above we obtain the following

$$T_2(t) = 2t^2 - 1, \quad T_3(t) = 4t^3 - 3t, \quad T_4(t) = 8t^4 - 8t^2 + 1.$$

The generating function of Chebyshev polynomials of first kind $T_n(t)$ is given as

$$F(z,t) = \frac{1 - tz}{1 - 2tz + z^2} = \sum_{n}^{\infty} = 0T_n(t)z^n, \quad t \in [-1, 1].$$

The Chebyshev polynomials of second kind is $U_n(t)$ and is given by

$$H(z,t) = \frac{1}{1 - 2tz + z^2} = \sum_{n}^{\infty} U_n(t)z^n, \quad t \in [-1, 1], \quad z \in A$$

where the first few coefficient of Chebyshev polynomials of second kind are given by as follows

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t).$$

$$(3)\quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, ...$$

here show the connection between first and second Chebyshev polynomials.

2. Preliminary and Definition

Recently, precisely in 2013, [15] investigated Sigmoid function in the space of univalent function and they obtained many interesting results. Since then many researchers have extensively studied activation function (see [27], [18] & [23]) and the literature therein. Let

$$y(s) = \frac{1}{1+e^{-s}}$$

be the Sigmoid function and the modified function be defined as

$$\gamma(s) = \frac{2}{1+e^{-s}}$$

(see for detail [15]).

Here we refer to a notation of $q-$operators, in essence $q-$difference operator and $q-$integral operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of $q-$calculus was introduced in [17] (see also [28] & [25]) and the literature therein and for detail about $q-$derivative operator see ([14], [17], [22], [28], [29], [30]).

For $q \in (0,1)$ the Jackson's $q-$derivative of a function $f \in A$ is given as

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 01 \end{cases}$$

and $D_q^2 f(z) = D_q(D_q f(z))$ [9].

From (4), we observe that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

where

$$[n]_q = \frac{1-q^n}{1-q}$$

which is sometimes referred to basic number n, if $q \to 1^{-1}$, $[n]_q \to n$

We find out that the following features hold

$$D_q f(z) + g(z) = (D_q f)(z) + (D_q g)(z)$$

$$D_q(f(z)q(z)) = g(q^{-1})(D_q f)(z) + f(qz)(D_q g)(z)$$

$$= g(qz)(D_q f)(z) + f(q^{-1}z)(D_q g)(z)$$

From (1), we have the following equation

$$(D_q f)(z) = (D_q^2 f)(q^{-1}z)$$

From (2) and (4) we observed that

$$(D_q g)(w) = \frac{g(qw) - g(q^{-1}w)}{(q - q^{-1})w}$$

$$(4) \quad = 1 - [2]_q a_2 w + [3]_q (2a_2^2 - a_3)w^2 - [4]_q (5a_2^3 - 5a_2a_3 + a_4)w^3 + \ldots$$

Motivated by the works of [1], [2], [11] & [14] and the present authors [5] we define the following

Definition 2.1. A function $f \in \sum$ given by (1) is said to be in the class $H^q_{\sum}(\phi(z,t), \gamma(s))$ if the following subordinate conditions holds:

$$(5) \quad (1-\gamma(s))(D_q f(z)) + \gamma(s)[1 + z(D_q f'(z))] \prec H(z,t) = \frac{1}{1-2tz+z^2} \quad \left(\frac{1}{2} < t < 1, \quad z \in U\right),$$

and

$$(6) \quad (1-\gamma(s))(D_q g(w)) + \gamma(s)[1 + z(D_q g'(w))] \prec H(w,t) = \frac{1}{1-2tw+w^2} \quad \left(\frac{1}{2} < t < 1, \quad w \in U\right),$$

where $\gamma(s) = \frac{2}{1+e^{-s}}$, is modified sigmoid function and $s \geq 0($ s is real) and $g = f^{-1}$. The class $H^q_{\sum}(s,t)$ is defined as follows:

Definition 2.2. A function $f \in \sum$ given by (1) is said to be in the class $H^q_{\sum}(\phi(z,t), \gamma(s))$ if the following subordinate conditions holds:

$$(7) \quad (1-\gamma(s))f'(z) + \gamma(s)[1 + zf''(z)] \prec H(z,t) = \frac{1}{1-2tz+z^2} \quad \left(\frac{1}{2} < t < 1, \quad z \in U,\right)$$

and

$$(8) \quad (1-\gamma(s))g'(w) + \gamma(s)[1 + zf''(w)] \prec H(w,t) = \frac{1}{1-2tw+w^2} \quad \left(\frac{1}{2} < t < 1, \quad w \in U\right),$$

for $g = f^{-1}$.

Definition 2.3 ([2] & [11]). If $w(z) = b_1 z + b_2 z^2 +, \cdots, b_1 \neq 0$ is analytic and satisfy $|w(z)| < 1$ on the unit disk E, then for each $0 < r < 1, |w'(z)| < 1$ and $\left|w(re^{i\theta})\right| < 1$ unless $w(z) = e^{i\theta} z$ for some real number θ.

2.1. Coefficients Bounds for the Class $H^q_{\sum}(\phi(z,t),\gamma(s))$.

Theorem 2.4. *Let* $f \in H^q_{\sum}(\phi(z,t),\gamma(s))$. *Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|4[(1+\gamma(s))[3]_q - [2]_q^2] + [2]_q^2\right|}} \tag{9}$$

$$|a_3| \leq \frac{4t^2}{[2]_q^2} + \frac{2t}{[1+\gamma(s)][3]_q} \tag{10}$$

$$|a_4| \leq \frac{10t^2}{[1+\gamma(s)][2]_q[3]_q} + \frac{8t^3+8t^2-2t-1}{[1+2\gamma(s)][4]_q} \tag{11}$$

Proof. Let $H^q_{\sum}(\phi(z,t),\gamma(s))$. Then from (5) and (6)

$$\begin{aligned}(1-\gamma(s))(D_qf(z)) + \gamma(s)[1+(D_qf'(z))]\\ = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots,\end{aligned} \tag{12}$$

and

$$\begin{aligned}(1-\gamma(s))(D_qg(w)) + \gamma(s)[1+(D_qg'(w))]\\ = 1 + U_1(t)v(w) + U_2(t)w^2(z) + \cdots.\end{aligned} \tag{13}$$

Define the functions $u(z)$ and $v(w)$ by

$$u(z) = c_1z + c_2z^2 + ... \tag{14}$$

$$v(w) = d_1w + d_2w^2 + ... \tag{15}$$

which are analytic in D with $u(0) = 0 = v(0)$ and $|u(z)| < 1$ and $|v(w)| < 1$ for all $z \in E$. It is well known that

$$|u(z)| = \left|c_1z + c_2z^2 + ...\right| < 1,$$
$$|v(w)| = \left|d_1w + d_2w^2 + ...\right| < 1$$

and then

$$|c_i| \leq 1,$$
$$|d_i| \leq 1.$$

It is evident from (12), (13), (14) and (15) we note that

$$(1-\gamma(s))(D_qf(z)) + \gamma(s)[1+(D_qf'(z))] = 1 + U_1(t)c_1z + (c_2U_1(t) + c_1^2U_2(t))z^2 +$$

$$(c_3U_1(t) + 2c_1c_2U_2(t) + c_1^3U_3(t))z^3 + \tag{16}$$

and

$$(1-\gamma(s))(D_qg(w)) + \gamma(s)[1+(D_qg'(w))] = 1 + U_1(t)d_1w + (d_2U_1(t) + d_1^2U_2(t))w^2 +$$

$$(d_3U_1(t) + 2d_1d_2U_2(t) + d_1^3U_3(t))w^3 + \tag{17}$$

We note from (16) and (17) that

$$[2]_q a_2 = c_1U_1(t) \tag{18}$$

$$[1 + \gamma(s)][3]_q a_3 = c_2U_1(t) + c_1^2U_2(t) \tag{19}$$

$$[1 + 2\gamma(s)][4]_q a_4 = c_3U_1(t) + 2c_1c_2U_2(t) + c_1^3U_3(t) \tag{20}$$

$$-[2]_q a_2 = d_1U_1(t) \tag{21}$$

$$[1 + \gamma(s)][2a_2^2 - a_3][3]_q a_3 = d_2U_1(t) + d_1^2U_2(t) \tag{22}$$

$$\begin{aligned} &-[1 + 2\gamma(s)][5a_2^3 - 5a_2a_3 + a_4][4]_q a_4 \\ &= d_3U_1(t) + 2d_1d_2U_2(t) + d_1^3U_3(t). \end{aligned} \tag{23}$$

From (18) and (21), we have

$$c_1 = -d_1U \tag{24}$$

$$a_2 = \frac{c_1U_1(t)}{[2]_q} = -\frac{c_1U_1(t)}{[2]_q} \tag{25}$$

and

$$2[2]_q^2 a_2^2 = U_1(t)[c_1^2 + d_1^2] \tag{26}$$

Adding (19) and (22) to get

$$2[1 + \gamma(s)][3]_q a_2^2 = [c_2 + d_2]U_1(t) + [c_1^2 + d_1^2]U_2(t) \tag{27}$$

Using (26) and (27), we have

$$\left[2(1 + \gamma(s))[3]_q - \frac{2U_2(t)}{U_1^2(t)}[2]_q^2\right] a_2^2 = [c_2 + d_2]U_1(t) \tag{28}$$

It is fairly well known from [8] that $|w(z)| < 1$ and $|v(z)| < 1$ then

$$|c_i| \le 1, \quad |d_i| \le 1 \ \ for \ \ i \in N \tag{29}$$

Using (28) and (29), we have from (28) that

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|4[(1 + \gamma(s))[3]_q - [2]_q^2] + [2]_q^2|}}.$$

Next, taking away (22) from (19) yields

$$2[1 + \gamma(s)][3]_q a_3 = 2[1 + \gamma(s)][3]_q a_2^2 + [c_2 + d_2]U_1(t) \tag{30}$$

We obtain from (30)

$$a_3 = a_2^2 + \frac{[c_2 - d_2]U_1(t)}{2[1+\gamma(s)][3]_q} \tag{31}$$

In view of (25) and (31)

$$a_3 = \frac{[c_1^2 + d_1^2]U_1^2(t)}{2[2]_q^2} + \frac{[c_2 - d_2]U_1(t)}{2[1+\gamma(s)][3]_q} \tag{32}$$

By considering (3) and (29), we obtain from (32) the desired inequality

$$|a_3| \leq \frac{4t^2}{[2]_q^2} + \frac{2t}{[1+\gamma(s)][3]_q}. \tag{33}$$

From (20) and (23) we get

$$a_4 = \frac{5}{2}a_2a_3 - \frac{5}{2}a_2^3 + \frac{[c_3 - d_3]U_1(t)}{2[1+2\gamma(s)][4]_q} + \frac{[2c_1c_2 - 2d_1d_2]U_1(t)}{2[1+2\gamma(s)][4]_q} + \frac{[c_1^3 - d_1^3]U_3(t)}{2[1+2\gamma(s)][4]_q} \tag{34}$$

From (25), (32) and (34) we get

$$a_4 = \frac{5c_1[c_1^2 + d_1^2]U_1^3(t)}{4[2]_q^3} + \frac{5c_1[c_2 - d_2]U_1^2(t)}{4[1+\gamma(s)][2]_q[3]_q} - \frac{5c_1^3U_1^3(t)}{2[2]_q^3} + \frac{[c_3 - d_3]U_1(t)}{2[1+2\gamma(s)][4]_q}$$

$$+ \frac{[2c_1c_2 - 2d_1d_2]U_1(t)}{2[1+2\gamma(s)][4]_q} + \frac{[c_1^3 - d_1^3]U_3(t)}{2[1+2\gamma(s)][4]_q} \tag{35}$$

Thus by Lemma 1, (3) and (29)

$$|a_4| \leq \frac{10t^2}{[1+\gamma(s)][2]_q[3]_q} + \frac{8t^3 + 8t^2 - 2t - 1}{[1+2\gamma(s)][4]_q}. \tag{36}$$

□

Corollary 2.5. *Let $f \in H^q_{\sum}(\phi(z,t), \gamma(0))$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|4[2[3]_q - [2]_q^2] + [2]_q^2\right|}} \tag{37}$$

$$|a_3| \leq \frac{4t^2}{[2]_q^2} + \frac{t}{[3]_q} \tag{38}$$

$$|a_4| \leq \frac{5t^2}{[2]_q[3]_q} + \frac{8t^3 + 8t^2 - 2t - 1}{3[4]_q} \tag{39}$$

Corollary 2.6. *Let $f \in H^q_{\sum}(\phi(z,t), \gamma(1))$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|[9.848[3]_q - 4[2]_q^2] + [2]_q^2\right|}} \tag{40}$$

$$|a_3| \leq \frac{4t^2}{[2]_q^2} + \frac{0.8123t}{[3]_q} \tag{41}$$

$$|a_4| \leq \frac{4.0617t^2}{[2]_q[3]_q} + \frac{0.2548(8t^3 + 8t^2 - 2t - 1)}{[4]_q} \tag{42}$$

Corollary 2.7. *Let $f \in H_{\sum}(\phi(z,t), \gamma(0))$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|2t^2 + 1|}} \tag{43}$$

$$|a_3| \leq t^2 + \frac{t}{3} \tag{44}$$

$$|a_4| \leq \frac{5}{6}t^2 + \frac{1}{12}(8t^8t^2 - 2t - 1) \tag{45}$$

Corollary 2.8. *Let $f \in H_{\sum}(\phi(z,t), \gamma(1))$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|13.544t^2 + 4|}} \tag{46}$$

$$|a_3| \leq t^2 + 0.2707t \tag{47}$$

$$|a_4| \leq 0.677t^2 + 0.0637[8t^8t^2 - 2t - 1] \tag{48}$$

Corollary 2.9. *Let $f \in H_{\sum}(\phi(z,1), \gamma(0))$. Then*

$$|a_2| \leq \frac{\sqrt{6}}{3} \tag{49}$$

$$|a_3| \leq \frac{3}{2} \tag{50}$$

$$|a_4| \leq \frac{23}{12} \tag{51}$$

Corollary 2.10. *Let* $f \in H_{\sum}(\phi(z,1),\gamma(1))$. *Then*

$$|a_2| \le \frac{2\sqrt{2}}{\sqrt{17.544}} \tag{52}$$

$$|a_3| \le 1.2707 \tag{53}$$

$$|a_4| \le 1.509 \tag{54}$$

2.2. **Fekete-Szego Inequalities for the Class** $H^q_{\sum}(\phi(z,t),\gamma(s))$**.**

Theorem 2.11. *Let* $f \in H^q_{\sum}(\phi(z,t),\gamma(s))$ *and* $\phi \in R$. *Then*

(55)

$$\left|a_3 - \phi a_2^2\right| \le \begin{cases} \frac{4t}{[1+\gamma(s)][3]_q}, & |\phi - 1| \le \beta \\ \\ \frac{16|\phi-1|t^3}{\left|4t^2[(1+\gamma(s))[3]_q - [2]_q^2] + [2]_q^2\right|}, & |\phi - 1| \ge \beta \end{cases}$$

where

$$\beta = \frac{\left|4t^2[(1+\gamma(s))[3]_q - [2]_q^2] + [2]_q^2\right|}{4(1+\gamma(s))[3]_q t^2} \tag{56}$$

Proof. From (25) and (32) we obtain

$$a_3 - \phi a_2^2 = \frac{(1-\phi)U_1^3(t)(c_2+d_2)}{2[(1+\gamma(s))[3]_q U_1^2(t) - [2]_q^2 U_2(t)]} + \frac{U_1(t)(c_2-d_2)}{(1+\gamma(s))[3]_q}$$

$$= U_1(t)\left[\left(h(\mu) + \frac{1}{2(1+\gamma(s))[3]_q}\right)c_2 + \left(h(\mu) - \frac{1}{2(1+\gamma(s))[3]_q}\right)d_2\right]$$

where

$$h(\Phi) = \frac{(1-\Phi)U_1^2(t)}{2[(1+\gamma(s))[3]_q U_1^2(t) - [2]_q^2 U_2(t)]}$$

Then in view of (3), we easily have

$$\left|a_3 - \phi a_2^2\right| \le \begin{cases} \frac{4t}{[1+\gamma(s)][3]_q}, & |\phi| \le \frac{1}{[1+\gamma(s)][3]_q} \\ \\ 8\,|h(t)|\,t, & |\phi - 1| \ge \frac{4t}{[1+\gamma(s)][3]_q} \end{cases}$$

□

Conclusion. The study discuss a subclass of bi-univalent analytic functions associated with the q-operator and the Sigmoid function via Chebyshev polynomials. Some estimates for the initial coefficients and the Fekete–Szegö functional were derived for the class $\sum$. Furthermore, by varying the parameters involved, several results were obtained, thereby enriching the existing literature.

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