Nigerian Journal of Mathematics and Applications
Volume **27**, (2018), 17 − 27. *Printed by Unilorin press*
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Parameter Estimation Methods for Fourier Regression Time Series Model

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Abstract

This paper present a multiple form of Fourier regression analysis. This model consist of one periodic response variable and several independent variables decompose into periodic components in order to have an uncorrelated predictors series. Ordinary least square and maximum likelihood methods of obtaining the parameters of the model was proved, the estimators are shown to be unbiased, variance of the estimators and variance of error term were derived and from the decomposed sum of square the coefficient of determination and adjusted coefficient was derived. The steps of test of hypothesis are stated and the performance of the error are checked based on Durbin Watson statistic to justify the consistency and reliability of the estimation methods.

1. Introduction

Periodic phenomena are primarily seen and observed in several fields. It is synchronized with daily, lunar and annual changes (Bliss, 1970). The periodic or cyclic character of many phenomena in natural environment is expressed in time and in space and these variations occur depending on daily, monthly, yearly or other cyclic changes. The analysis of relationship between periodic variations has been done using so many methods (Rawlings *et al.*, 1998). In these methods,

Received: 22/07/2017, Accepted: 15/10/2017, Revised: 13/11/2017.
2015 *Mathematics Subject Classification.* 62H12 & 62M10.
Key words and phrases. Fourier series; Regression; Periodic; Ordinary least square and maximum likelihood
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cautions are not put in place to account for the phase, amplitude and periodic components, even the analysis of periodic variables has been done by discussing the periodic changes in the response variables with respect to time (see, Bliss 1958, Korngold 1964, Popi*ński* 1999, Bloomfield 2000, Cobanovic *et al.,* 2006, Xiajian *et al.,* 2014 and Taiwo 2017).

This paper is used to propose a model that will accommodate several Fourier decomposed independent variables and this will be done using Fourier series and classical regression analysis steps. The theoretical aspect of the model and estimation of its parameters will be discussed based on ordinary least square and maximum likelihood methods. Significance of the estimated parameters will be as well discussed.

2. Methodology

The following are the theory involved in the article.

2.1 Fourier Regression Analysis

Given a simple Fourier deterministic model as:

$$y = \rho\cos(\omega t - \theta) \tag{1}$$

where ρ is the amplitude, ω is the frequency and θ is the phase.

Using the compound angle formula

$$\cos(A - B) = cosAcosB + sinAsinB$$

We rewrite (1) as

$$y = \rho\cos\theta cos(\omega t) + \rho\sin\theta sin(\omega t)$$

$$y = \beta\cos(\omega t) + \beta^{*}\sin(\omega t) \tag{2}$$

where $\beta = \rho cos\theta$, $\beta^{*} = \rho sin\theta$ and $\beta^2 + \beta^{*2} = \rho^2$.

Based on (2) a Fourier multiple regression model can be expressed

$$y_t = \beta_0 + \sum_{j=0}^{p} \beta_j\cos(\omega_j t) + \sum_{j=0}^{p} \beta_j^{*}\sin(\omega_j t) + \varepsilon_t, \qquad \begin{array}{l} j = 1, \ldots, p \\ t = 1, \ldots, T \end{array} \tag{3}$$

where $\omega_j = \frac{2\pi j}{n}$ and n is the length of observation.

By substituting $t = X_t$ in (3), we have a multiple form of Fourier regression model given as

$$y_t = \beta_0 + \sum_{j=1}^{p} \beta_j cos\omega X_t + \sum_{j=1}^{p} \beta_j^{*} sin\omega X_t + \varepsilon_t, \qquad \begin{array}{l} j = 1, \ldots, p \\ t = 1, \ldots, T \end{array} \tag{4}$$

where X_t are the predictors, y_t is the dependent variable and $\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$ is the error term.

We can as well express the predictors in their lags and this is given as

$$y_t = \beta_0 + \sum_{j=1}^{p} \beta_j cos\omega X_{t-i} + \sum_{j=1}^{p} \beta_j^* sin\omega X_{t-i} + \varepsilon_t, \qquad \begin{array}{l} j = 0, \dots, p \\ i = 1, \dots, T \end{array} \tag{5}$$

where X_{t-i} are the predictors, y_t is the dependent variable and $\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$ is the error term.

2.2.1 Methods of Parameter Estimation

The parameter estimation involves ordinary least squares and maximum likelihood estimation methods.

2.2.2. Ordinary Least Squares Estimation Method

Based on the multiple form of Fourier regression model in (4)

$$y_t = \beta_0 + \beta_j cos\omega X_t + \beta_j^* sin\omega X_t + \varepsilon_t$$

$$\varepsilon_t = y_t - \beta_0 - \beta_j cos\omega X_t - \beta_j^* sin\omega X_t$$

The residual sum of square is

$$\sum (\varepsilon_t)^2 = \sum \left(y_t - \beta_0 - \beta_j \cos\omega X_t - \beta_j^* \sin\omega X_t\right)^2 \tag{6}$$

Firstly we will differentiate (6) with respect to β_0, we have

$$\frac{\partial \sum (\varepsilon_t)^2}{\partial \beta_0} = -2 \sum (y_t - \beta_0 - \beta_j \cos\omega X_t - \beta_j^* \sin\omega) X_t$$

$$= -2 \sum (y_t - \beta_0 - \beta_j \cos\omega X_t - \beta_j^* \sin\omega \mathrm{t} X_t) \tag{7}$$

Also differentiate equation (6) with respect to β_j

$$\frac{\partial \sum (\varepsilon_t)^2}{\partial \beta_j} = 2 \sum (y_t - \beta_0 - \beta_j \cos\omega X_t - \beta_j^* \sin\omega X_t)(-\cos\omega X_t)$$

$$= -2 \sum (\cos\omega X_t)(y_t - \beta_0 - \beta_j \cos\omega X_t - \beta_j^* \sin\omega X_t) \tag{8}$$

Lastly differentiate (6) with respect to β_j^*

$$\frac{\partial \sum (e_t)^2}{\partial \beta_j^*} = 2 \sum (y_t - \beta_0 - \beta_j \cos\omega X_t - \beta_j^* \sin\omega) X_t (-\sin\omega X_t)$$

$$= -2 \sum (\sin\omega X_t)(y_t - \beta_0 - \beta_j \cos\omega X_t - \beta_j^* \sin\omega X_t) \tag{9}$$

Setting (7), (8) and (9) to zero gives

$$= \Sigma y_t - n\beta_0 - \beta_j \Sigma \cos\omega X_t - \beta_j^* \Sigma \sin\omega X_t$$

$$= \Sigma(\cos\omega X_t y_t) - \beta_0 \Sigma(\cos\omega X_t) - \beta_j \Sigma(cos^2\omega X_t) - \beta_j^* \Sigma(\cos\omega X_t \sin\omega X_t)$$

$$= \Sigma(\sin\omega X_t y_t) - \beta_0 \Sigma(\sin\omega X_t) - \beta_j \Sigma(\cos\omega X_t \sin\omega X_t) - \beta_j^* \Sigma sin^2\omega X_t$$

$$\left.\begin{array}{c}\Sigma y_t = n\beta_0 + \beta_j \Sigma\cos\,\omega X_t + \beta_j^* \Sigma\ \sin\omega X_t \\ \Sigma\left(\cos\omega X_t y_t\right) = \beta_0 \Sigma\left(\cos\omega X_t\right) + \beta_j \Sigma\left(cos^2\omega X_t\right) + \beta_j^*\ \Sigma(\cos\omega X_t \sin\omega X_t) \\ \Sigma\left(\sin\omega X_t y_t\right) = \beta_0 \Sigma\left(\sin\omega X_t\right) + \beta_j \Sigma\left(\cos\omega X_t \sin\omega X_t\right) + \beta_j^*\ \Sigma sin^2\omega X_t)\end{array}\right\} \tag{10}$$

Transforming (10) into matrix form, we have

$$\begin{pmatrix}\sum(y_t) \\ \sum(\cos\,\omega X_t y_t) \\ \sum(\sin\omega X_t y_t)\end{pmatrix} = \begin{pmatrix} n & \sum(\cos\,\omega X_t) & \sum(\sin\,\omega X_t) \\ \sum(\cos\,\omega X_t) & \sum(\cos^2\omega X_t) & \sum(\cos\omega \mathrm{t}\ \sin\omega X_t) \\ \sum(\sin\,\omega X_t) & \sum(\cos\omega\ X_t \sin\omega) X_t & \sum(\sin^2\omega X_t)\end{pmatrix}\begin{pmatrix}\beta_0 \\ \beta_j \\ \beta_j^*\end{pmatrix}$$

hence

(11)

$$\begin{pmatrix}\beta_0 \\ \beta_j \\ \beta_j^*\end{pmatrix} = \begin{pmatrix} n & \sum(\cos\,\omega X_t) & \sum(\sin\,\omega X_t) \\ \sum(\cos\,\omega X_t) & \sum(\cos^2\omega X_t) & \sum(\cos\omega \mathrm{t}\ \sin\omega X_t) \\ \sum(\sin\,\omega X_t) & \sum(\cos\omega\ X_t \sin\omega) X_t & \sum(\sin^2\omega X_t)\end{pmatrix}^{-1}\begin{pmatrix}\sum(y_t) \\ \sum(\cos\,\omega X_t y_t) \\ \sum(\sin\omega X_t y_t)\end{pmatrix}$$

From the above matrix, we have the estimated parameter.
By representing (11) with

$$\hat{W} = \left(G^{|}G\right)^{-1} G^{|} y \tag{12}$$

where $\widehat{?(} = \begin{pmatrix}\beta_0 \\ \beta_j \\ \beta_j^*\end{pmatrix}$, $\left(G^{|}G\right)^{-1} = \begin{pmatrix} n & \sum(\cos\,\omega X_t) & \sum(\sin\,\omega X_t) \\ \sum(\cos\,\omega X_t) & \sum(\cos^2\omega X_t) & \sum(\cos\omega \mathrm{t}\ \sin\omega X_t) \\ \sum(\sin\,\omega X_t) & \sum(\cos\omega\ X_t \sin\omega) X_t & \sum(\sin^2\omega X_t)\end{pmatrix}^{-1}$
and

$$G^{|}y = \begin{pmatrix}\sum(y_t) \\ \sum(\cos\,\omega X_t y_t) \\ \sum(\sin\omega X_t y_t)\end{pmatrix}$$

2.2.3 Distributional Properties of the Fourier Regression Estimator

$$\hat{W} = \left(G'G\right)^{-1} G' y^*$$

$$= \left(G'G\right)^{-1} G'\left(GW + \varepsilon\right)$$

$$\hat{W} = \left(G'G\right)^{-1} G'\ GW + \varepsilon$$

$$\hat{W} = \ W + G'\varepsilon$$

Taking expectation, we have

$$\hat{W} = \ [W] + G'E\left[\varepsilon\right]$$

$$\hat{W} = \ W + 0$$

Then

$$\hat{W} = \ W \tag{13}$$

This shows that the estimators are unbiased.

2.2.3 Dispersion of $\hat{W}$

(14) $$D(\hat{W}) = var(\hat{W}) = E[\hat{W} - W]'[\hat{W} - W]$$

since

$$\hat{W} - W = \left(G'G\right)^{-1}G'y^* - W$$
$$\hat{W} - W = \left(G'G\right)^{-1}G'\ \varepsilon$$

Therefore,

$$D\left(\hat{W}\right) = E\left[\hat{W} - W\right]'\left[\hat{W} - W\right]$$
$$= E\left[\left(G'G\right)^{-1}G'\ \varepsilon\right]$$
$$= E\left[\left(G'G\right)^{-1}G'\ \varepsilon\varepsilon'\left(G'G\right)^{-1}\right]$$
$$= \left(G'G\right)^{-1}G'\epsilon\left[\varepsilon\varepsilon'\right]G\left[G'G\right]^{-1}$$
$$= \left(G'G\right)^{-1}G'\sigma_e^2\quad G\left[G'G\right]^{-1}$$
$$= \left[G'G\right]^{-1}\sigma_e^2$$

(15)
$$D\left(\hat{W}\right) = \begin{pmatrix} n & \sum(\cos\omega X_t) & \sum(\sin\omega X_t) \\ \sum(\cos\omega X_t) & \sum(\cos^2\omega X_t) & \sum(\cos\omega\text{t}\ \sin\omega X_t) \\ \sum(\sin\omega X_t) & \sum(\cos\omega\ X_t\sin\omega)X_t & \sum(\sin^2\omega X_t) \end{pmatrix}\sigma_\varepsilon^2$$

where σ_ε^2 is the variance of the error term.

2.2.4 Variance of the Error Term

Given that

$$y = GW + \varepsilon$$
$$\varepsilon = y - GW$$
$$\varepsilon = [y - W\left[\left(G'G\right)^{-1}G'y^*\right]$$
$$\varepsilon = y\left[1 - \left(G'G\right)^{-1}G'\right]$$
$$\varepsilon = y\,[M]$$
$$\varepsilon = (GW + \varepsilon)\,M$$
$$\varepsilon = MGW + M\varepsilon$$
$$\varepsilon = WG\left[1 - \left(G'G\right)^{-1}G'\right] + M\varepsilon$$
$$\varepsilon = WG - WG\left(G'G\right)^{-1}G' + M\varepsilon$$
$$\varepsilon = M\varepsilon$$

If $\varepsilon' = (M\varepsilon)'$

$$\varepsilon\varepsilon' = M'\varepsilon'\text{M}\varepsilon$$
$$= \varepsilon' M\varepsilon\,[law\ of\ Idempotent]$$
$$\varepsilon\varepsilon' = trace\ \left[M\varepsilon\varepsilon'\right]$$
$$\text{E}[\varepsilon'\varepsilon] = tr\left[1-(G'G)^{-1}G'\right]E\,[\varepsilon\varepsilon']$$
$$\text{E}[\varepsilon'\varepsilon] = tr\,[I_n - I_k]\,\sigma_\varepsilon^2$$
$$\sigma_\varepsilon^2 = \frac{\text{E}\,[\varepsilon'\varepsilon]}{n-k}$$

or

$$\sigma_\varepsilon^2 = \frac{y'y - WG'y}{n-k}$$

Hence

$$\sigma^2\varepsilon = y'y^* - \begin{pmatrix} \beta_0 & \beta_j & \beta_j^* \end{pmatrix}\begin{pmatrix} \sum y_t \\ \sum cos\omega x_t y_t \\ \sum sin\omega x_t y_t \end{pmatrix}/n-k \tag{16}$$

2.2.5 Variance of $\left(\widehat{\boldsymbol{W}}\right)$

$$\text{V}\left(\hat{W}\right) = E\left[W - E\left(\hat{W}\right)\right]^2$$

From the unbiasedness of the estimator

$$\hat{W} = W + \left(G'G\right)^{-1}G'\varepsilon$$
$$\hat{W} - W = \left(G'G\right)^{-1}G'\varepsilon$$
$$V\left(\hat{W}\right) = E\left[\left(G'G\right)^{-1}G'\varepsilon\right]'\left[\left(G'G\right)^{-1}G'\varepsilon\right]$$
$$V\left(\hat{W}\right) = E\,[\varepsilon'\varepsilon]\,(G'G)^{-1}G'G\,(G'G)^{-1}$$
$$= E\left[\varepsilon'\varepsilon\right]\left(G'G\right)^{-1}$$
$$V\left(\hat{W}\right) = \sigma_e^2 \quad (G'G)^{-1}$$

$$V\left(\hat{W}\right) = \left(y'y^* - \begin{pmatrix} \beta_0 & \beta_j & \beta_j{}^* \end{pmatrix}\begin{pmatrix} \sum y_t \\ \sum \text{cos}\omega x_t y_t \\ \sum \text{sin}\omega x_t y_t \end{pmatrix}/(n-k)\right)\times$$
$$\begin{pmatrix} n & \sum(\text{cos}\omega X_t\ ) & \sum(\text{sin}\omega X_t\ ) \\ \sum(\text{cos}\omega X_t\ ) & \sum\left(\cos^2\omega X_t\right) & \sum(\text{cos}\omega\text{t sin}\omega X_t) \\ \sum(\text{sin}\omega X_t\ ) & \sum(\text{cos}\omega\ X_t\text{sin}\omega)\,X_t & \sum(\sin^2\omega X_t) \end{pmatrix}^{-1} \tag{17}$$

2.2.6 Decomposition of Sum of Square

This is done by partitioning the sum of square as follows

$$y_t - \overline{y} = \hat{y}_t - \overline{y} + y_t - \hat{y} \tag{18}$$

$$\sum (y_t - \overline{y})^2 = \sum (\hat{y}_t - \overline{y})^2 + \sum (y_i - \hat{y})^2$$
$$SST = SSE + SSR$$
$$SST = \sum (y_i - \overline{y})^2 = \sum {y_i}^2 - n\overline{y}^2$$
$$\text{SST} = \text{y}' y - n\overline{y}^2$$
$$SSE = \sum (\hat{y}_t - \overline{y})^2$$
$$\sum {y_t}^2 = W' G' y$$

Therefore

$$\sum {y_t}^2 = W' G' - n\overline{y}^2$$

Hence

$$SSE = \begin{pmatrix} \beta_0 & \beta_j & {\beta_j}^* \end{pmatrix} \begin{pmatrix} \sum y_t \\ \sum \text{cos} x_t y_t \\ \sum \text{sin} x_t y_t \end{pmatrix} - n\overline{y}^2$$

So

$$SSR = \sum (\hat{y}_t - - \overline{y})^2$$
$$= y' y - \beta' G' y$$
$$= y' y - \begin{pmatrix} \beta_0 & \beta_j & {\beta_j}^* \end{pmatrix} \begin{pmatrix} \sum y_t \\ \sum \text{cos} x_t y_t \\ \sum \text{sin} x_t y_t \end{pmatrix}$$
$$R^2 = \frac{\hat{W}' G' G W - n\overline{y}^2}{y' y - n\overline{y}^2}$$
$$= \frac{\hat{W}' G' \hat{y} - n\overline{y}^2}{y' y - n\overline{y}^2}$$
$$= \frac{\left(y' y - n\overline{y}^2\right) - \varepsilon' \varepsilon}{y' y - n\overline{y}^2}$$
$$= 1 - \frac{\varepsilon' \varepsilon}{y' A y}$$

where $A = 1 + \frac{n\overline{y}^2}{y' y}$
Thus

$$R^2 = 1 - \frac{SSE}{SST}$$

$$R^2 = 1 - \frac{\begin{pmatrix} \beta_0 & \beta_j & \beta_j{}^* \end{pmatrix} \begin{pmatrix} \sum y_t \\ \sum \mathrm{cos} x_t y_t \\ \sum \mathrm{sin} x_t y_t \end{pmatrix} - n\overline{y}^2}{y'y - \begin{pmatrix} \beta_0 & \beta_j & \beta_j{}^* \end{pmatrix} \begin{pmatrix} \sum y_t \\ \sum \mathrm{cos} x_t y_t \\ \sum \mathrm{sin} x_t y_t \end{pmatrix}} \tag{19}$$

and

$$\overline{R}^2 = \frac{1}{n-k}[nR^2 - k] \tag{20}$$

2.2.7 Maximum Likelihood Estimation Method

If the error random variables (ε_t) are assumed to be independent and normally distributed with mean zero, then the maximum likelihood estimates will be obtained as follows

$$L = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_t - \beta_0 - \beta_j cos\omega X_t - \beta_j^* sin\omega X_t}{\sigma}\right)^2} \tag{21}$$

Based on

$$\varepsilon_t = y_t - \beta_0 - \beta_j cos\omega X_t - \beta_j^* sin\omega X_t$$

the likelihood function with respect to the error term becomes

$$L = (\sigma\sqrt{2}\pi)^{-1} e^{-\frac{1}{2}\left(\frac{y_1 - \beta_o - \beta_j \cos\ \omega X_1 - \beta_j^* \ \sin\omega X_1}{\sigma}\right)^2}$$

$$\times(\sigma\sqrt{2}\pi)^{-1} e^{-\frac{1}{2}\left(\frac{y_2 - \beta_o - \beta_j \cos\ \omega X_2 - \beta_j^* \ \sin\omega X_2}{\sigma}\right)^2} \times \cdots$$

$$\times(\sigma\sqrt{2}\pi)^{-1} e^{-\frac{1}{2}\left(\frac{y_m - \beta_o - \beta_j \cos\ \omega X_m - \beta_j^* \ \sin\omega X_m}{\sigma}\right)^2} \tag{22}$$

Taking the log of (22) and linearize, we have

$$lnL = -mln\ (\sigma\sqrt{2}\pi) - \frac{1}{2}\sum_{t=1}^{m}\left(\frac{y_t - \beta_0 - \beta_j cos\omega X_t - \beta_j^* sin\omega X_t}{\sigma}\right)^2 \tag{23}$$

Differentiate (23) with respect to β_0 and equate to zero

$$= \frac{-2}{2\sigma^2}\sum_{t=1}^{m}\left(y_t - \widehat{\beta}_0 - \widehat{\beta}_j cos\omega X_t - \widehat{\beta}_j^* sin\omega X_t\right)(-1) = 0$$

$$\sum_{t=1}^{m}(y_t - \widehat{\beta}_0 - \widehat{\beta}_j cos\omega X_t - \widehat{\beta}_j^* sin\omega X_t) = 0$$

$$\sum y_t - m\widehat{\beta}_0 - \widehat{\beta}_j \sum cos\omega X_t - \sum \widehat{\beta}_j^* sin\omega X_t = 0$$

$$m\widehat{\beta}_0 = \sum y_t + \widehat{\beta}_j \sum cos\omega X_t + \sum \widehat{\beta}_j^* sin\omega ??_t$$

Divide both sides by m, we have

$$\widehat{\beta}_0 = \frac{\sum y_t}{m} - \widehat{\beta}_j \frac{\sum \cos\omega X_t}{m} - \widehat{\beta}_j^* \ \frac{\sum \sin\omega X_t}{m} \tag{24}$$

Also, differentiate (23) with respect to β_j and equate to zero

$$= \frac{-2}{2\sigma^2} \sum_{t=1}^{m} \left(y_t - \widehat{\beta}_0 - \widehat{\beta}_j cos\omega X_t - \widehat{\beta}_j^* sin\omega X_t \right) (-cos\omega X_t)$$

$$= \sum_{t=1}^{m} (y_t - \widehat{\beta}_0 - \widehat{\beta}_j cos\omega X_t - \widehat{\beta}_j^* sin\omega X_t)(cos\omega X_t) = 0$$

$$= \sum (cos\omega X_t) y_t - \widehat{\beta}_0 \sum cos\omega X_t - \widehat{\beta}_j \sum (cos\omega X_t)^2 - \widehat{\beta}_j^* \sum cos\omega X_t sin\omega X_t = 0$$

$$\widehat{\beta}_j \sum (cos\omega X_t)^2 = \sum (cos\omega X_t) y_t - \widehat{\beta}_0 \sum cos\omega X_t - \widehat{\beta}_j^* \sum cos\omega X_t sin\omega X_t$$

(25)

$$\widehat{\beta}_j = \sum (cos\omega X_t) y_t - \widehat{\beta}_0 \sum cos\omega X_t - \widehat{\beta}_j^* \sum cos\omega X_t sin\omega X_t / \sum (cos\omega X_t)^2$$

By differentiating (23) with respect to β_j^* as well and equate to zero, we will have

$$= \frac{-2}{2\sigma^2} \sum_{t=1}^{m} \left(y_t - \widehat{\beta}_0 - \widehat{\beta}_j cos\omega X_t - \widehat{\beta}_j^* sin\omega X_t \right) (-sin\omega X_t)$$

$$= \sum_{t=1}^{m} (y_t - \widehat{\beta}_0 - \widehat{\beta}_j cos\omega X_t - \widehat{\beta}_j^* sin\omega X_t)(sin\omega X_t) = 0$$

$$= \sum (sin\omega X_t) y_t - \widehat{\beta}_0 \sum sin\omega X_t - \widehat{\beta}_j \sum cos\omega X_t sin\omega X_t - \widehat{\beta}_j^* \sum (sin\omega X_t)^2 = 0$$

$$\widehat{\beta}_j^* \sum (sin\omega X_t)^2 = \sum (sin\omega X_t) y_t - \widehat{\beta}_0 \sum sin\omega X_t - \widehat{\beta}_j \sum cos\omega X_t sin\omega X_t$$

(26) $\widehat{\beta}_j^* = \sum (sin\omega X_t) y_t - \widehat{\beta}_0 \sum sin\omega X_t - \widehat{\beta}_j \sum cos\omega X_t sin\omega X_t / \sum (sin\omega X_t)^2$

The results in equations (24), (25) and (26) gives the parameters estimates.

2.2.8 Test of Hypothesis

For individual parameter, we will carry-out test of hypothesis as follows

$$H_0 : \widehat{W_j} = 0$$

$$H_1 : \widehat{W_j} \neq 0$$

$$t = \frac{\widehat{W}_j}{\sqrt{\sigma_\varepsilon{}^2 C_{jj}}} = \frac{\widehat{W}_j}{S_\varepsilon\left(\widehat{W}_j\right)} \sim t_{\propto/2},\ (n-k-1) \tag{27}$$

where C_{jj} is the diagonal element of $(G'G)^{-1}$ corresponding to $\widehat{W}_j$, α is the level of significance and $(n-k-1)$ degree of freedom.
For the joint hypothesis, the steps below will be followed

$$H_0:\ \beta_0 = \beta_1 = \beta_1^* = 0$$
$$H_1: \beta_0 = \beta_1 = \beta_1^* \neq 0$$

$$F = \frac{SSR/k}{SSE/(n-k-1)} = \frac{MSR}{MSE} \tag{28}$$

with $(k-1)$ *and* $(n-k-1)$ degree of freedom.

2.2.9 Error Term Performance based on Durbin Watson Statistic

If ε_t is the residual associated with the observation at time t, then the test statistic is

$$d = \frac{\sum_{t=2}^{T}(\varepsilon_t - \varepsilon_{t-1})^2}{\sum_{t=1}^{T}\varepsilon_t^2} \tag{29}$$

where T is the number of observations. Note that if the sample is lengthy, then this can be linearly mapped to the Pearson correlation of the time-series data with its lags (Durbin and Watson 1950). Since d is approximately equal to $2(1r)$, where r is the sample autocorrelation of the residuals, $d = 2$ indicates no autocorrelation. The value of d always lies between 0 and 4. If the DurbinWatson statistic is substantially less than 2, there is evidence of positive serial correlation. As a rough rule of thumb, if DurbinWatson is less than 1.0 there may be cause for alarm. Small values of d indicate successive error terms are on average close in value to one another or positively correlated. If $d > 2$ successive error terms are on average much different in value from one another, i.e., negatively correlated (Durbin and Watson 1951) where this can imply an underestimation of the level of statistical significance (Gujarati and Porter 2009).

3. Discussion and Conclusion

This research work is used to present the framework of a multiple form of Fourier regression analysis. This model consist of one periodic response variable and several independent variables decompose into periodic components in order to have a near uncorrelated predictors series. Ordinary least square and maximum likelihood methods of obtaining the parameters of the model were proved. The estimators are shown to be unbiased, variance of the estimators and variance of error term were derived and from the decomposed sum of square the coefficient of determination and adjusted coefficient was derived. The steps of test of hypothesis are stated and the performance of the error are checked based on Durbin Watson Statistic. Conclusively, a multiple form of parameter estimation method for Fourier regression time series analysis was derived and its distributional estimator properties were presented. The steps of hypothesis testing and error diagnosis were also derived to justify the consistency and reliability of the estimation methods.

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