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Optimal Control Analysis of an SIRS Epidemic Model with the Asymptotic Transmission

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Abstract

In this paper, we developed and analysed a SIRS(Susceptible-Infected-Recovered-Susceptible) epidemic model that captures the asymptotic transmission in the presence of two control interventions (vaccination and treatment). Non-linear deterministic system of differential equations was used to capture the asymptotic transmission of the SIRS epidemic model and the basic reproduction number, R_0, with the two control interventions of the model was computed. Optimal control theory was applied using the Pontryagin's Maximum Principle (PMP) to investigate optimal strategies for controlling the spread of the disease using vaccination and treatment as the system control variables. The result predicted that both treatment and vaccination strategies are very effective control strategies to combat the transmission of any disease but in areas of limited resources treatment only or vaccination only can be applied to reduce the morbidity and mortality of any disease. Finally, numerical simulations were performed and graphical results were presented to illustrate the analytical solution.

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1. Introduction

Mathematical models have become important tools for analyzing and predicting the spread and control of infectious and non-infectious diseases [37]. The process of model formulation process clarifies assumptions, variables, and parameters. Mathematical models provide conceptual results such as thresholds like the basic reproduction numbers, contact numbers, and replacement numbers[37]. Mathematical models and computer simulations are useful experimental tools for building and testing theories, assessing quantitative conjectures, answering specific questions, determining sensitivities to changes in parameter values, and estimating key parameters from data[37,38]. Mathematical models are used in comparing, planning, implementing, evaluating, and optimizing various detection, prevention, therapy, and control programs [37,38].

A mathematical model known as the SIR epidemic model was first proposed by Kermack and McKendrick in 1927, which happens to be introduction to epidemic modelling [47]. Since then, there have been various development of different kinds of model structures like SIS, SEIR,SERS, SIRS and many others. Review of the literatures [8, 17, 21, 22, 23, 36, 38, 41, 72] show the rapid growth of epidemiology modeling. The recent models have involved aspects such as passive immunity, gradual loss of vaccine and disease-acquired immunity, stages of infection, vertical transmission, disease vectors, macro-parasitic loads, age structure, social and sexual mixing groups, spatial spread, vaccination, quarantine, and chemotherapy. Special models have been formulated for diseases such as measles, rubella, chickenpox, whooping cough, diphtheria, smallpox, malaria, onchocerciasis, filariasis, rabies, gonorrhea, herpes, syphilis, and HIV/AIDS. Most of the works done have been shown in the books on epidemiology modeling [1, 2, 3, 4, 5, 6, 7, 9, 15, 16, 14, 18, 19, 20, 28, 21, 33, 38, 39,48 50, 52, 55, 59, 61, 63, 66, 67]. There are several mathematical models for the spread of infectious disease in populations which have been analyzed mathematically and applied to specific diseases [36]. Many studies have been carried out on epidemic models and reviews on theoretical developments [10, 24, 32, 42, 41, 3, 12, 14, 20, 36, 11, 13, 43, 56, 53, 62, 65, 68, 69, 63]. A system of differential equations for measles epidemic with vaccination is derived in [29]. [55] studied the consequences of vaccination in a discrete-time model for the spread of periodic diseases.

Optimal control theory is another area of mathematics that is applied extensively to make decisions on the control of the spread of infectious and non-infectious diseases [34]. It has been discovered that the application of optimal control theory is a powerful mathematical tool that can be used to make decisions involving complex biological situation [34]. It is often used to predict the control of the spread of most diseases for which either vaccine, treatment or any other control variables are available [34]. Optimal control theory has been applied to a set of epidemiological models in an attempt to obtain the most effective control strategy

to minimize the number of individuals who become infected in the course of an epidemic in the presence of both treatment and vaccination as control measures [30]. [72] presented a model in relation to [30] but concentrated on a SIR model using only vaccination as the control intervention. [47] applied optimal control theory to determine the optimal treatment strategy for the administration of the antiretroviral drug (Reverse Transcriptase Inhibitors) in HIV-positive individuals. [33] also used optimal control theory to determine the condition for the elimination of tumor cells in an individual under treatment for Cancer.[71] worked on the optimal control of vaccination and treatment for a SIR epidemiological model. In their work, they considered a SIR model with variable population size and formulate an optimal control problem to the model with vaccination and treatment as controls. They showed that the disease-free equilibrium is globally asymptotically stable if the basic reproduction number(R_0) is less than unity while the endemic equilibrium exists and it is globally stable if $R_0 > 1$. They also used the Pontryagin's Maximum Principle to characterize the optimal levels of the two controls. [36] worked on stability analysis and optimal control of a SIR epidemic model with vaccination. Their work focuses on the study of a nonlinear mathematical SIR epidemic model with a vaccination program. They have discussed the existence and the stability of both the disease-free and endemic equilibriums. [59] worked on the rich dynamics of a SIR epidemic model and the work aims to study a SIR epidemic model with an asymptotically homogeneous transmission function. The stability of the disease-free and the endemic equilibriums and carried out the numerical simulations were addressed. Implications of the analytical and numerical findings are discussed critically. In this paper, we extended the the work of Pathak et al. [59] by developing and investigating a deterministic non-linear system of differential equation under the influence of the asymptotic transmission, the reproduction number of the model is carried out and the mathematical optimal control theory is applied to investigate the optimal strategies of two control variables(vaccination and treatments).The aim here is to predict the impact of the two control interventions in the presence of asymptotic transmission for a disease with SIRS model structure (e.g. Ebola, Influenza, Malaria, Zika, Cholera etc).

The paper is organized as follows: in section 2, we present the model formulation. In section 3, we discuss the mathematical analysis of the model and the controlled reproduction number. In section 4, we present an optimal control problem subject to the controlled dynamics(SIRS model) with its characterization and the derivation of the optimality system using Pontryagin's Maximum Principles(PMP). While, in section 5, we present the resulting solution of our optimality system numerically with numerical simulation. In section 6, we discussed the conclusion and possible extensions.

2. Model Formulation

We consider an SIRS epidemic model with rich dynamics and variable total population. The total population is $N = S(t) + I(t) + R(t)$. The population of the susceptible is given by the recruitment rate into the population(at a per capita rate b). It is also generated by the rate at which recovered individuals looses immunity and return to the susceptible class at a rate. It is reduced by infection after contacts with infected individual at a per capita rate of transmission $f(S, I) = \frac{kSI}{(1+\alpha S+\beta I)}$ and by natural death(at a rate d). It is also been reduced by vaccination(at a rate u_1), where $\alpha$and β are the parameters which measure the effects of sociological, psychological or other mechanisms. The rate of change of the population of susceptible Individual is given by:

$$\frac{dS}{dt} = b - dS - \frac{kSI}{(1 + \alpha S + \beta I)} + \gamma R + \varpi I - u_1 S \tag{1}$$

Similarly the rate of change of the population of infected Individual is given by:

$$\frac{dI}{dt} = \frac{kSI}{(1 + \alpha S + \beta I)} - (d + \varpi + \mu + \rho)I - u_2 I \tag{2}$$

, where u_2 is the treatment rate and μ is the natural recovery rate, ρ is the disease induced death rate and ϖ is the constant rate at which the infected become susceptible. The rate of change of the population of recovered Individual is given by:

$$\frac{dR}{dt} = \mu I - (d + \gamma)R + u_1 S + u_2 I \tag{3}$$

, where γ is the rate at which recovered individual looses immunity and return to the susceptible class. By adding altogether gives $N = S(t) + I(t) + R(t)$ implies

$$\frac{dN}{dt} = b - dN - \rho I. \tag{4}$$

It is observed that the total population size N is a variable in the absence of the disease, the total human population size, N approaches a carrying capacity $\frac{b}{d}$. We described the model variables with constant controls u_1 and u_2 and parameters below:

$$\begin{array}{rcl} \frac{dS}{dt} & = & b - dS - \frac{kSI}{(1+\alpha S+\beta I)} + \gamma R + \varpi I - u_1 S, \\ \frac{dI}{dt} & = & \frac{kSI}{(1+\alpha S+\beta I)} - (d + \varpi + \mu + \rho)I - u_2 I, \\ \frac{dR}{dt} & = & \mu I - (d + \gamma)R + u_1 S + u_2 I. \end{array} \tag{5}$$

which adding together gives $N = S + I + R$ implies:

$$\frac{dN}{dt} = b - dN - \rho I.$$

We discovered that the total population size N is a variable population in which in the absence of the disease, the total population is constant. The SIRS epidemic model (5) will be analyzed in a biologically feasible region as follows. This region should be feasible for the total populations. If b is a constant then the feasible region for (5) in Ω is

$$\Omega = \{(S, I, R) \in \mathbb{R}^3_+ : S + I + R \leq \frac{b}{d}\}. \tag{6}$$

Since by (5), $\limsup_{t\to\infty} S(t) \leq \frac{b}{d}$, thus the global attractor of (5) is contain in Ω. Thus the dynamical behaviour of (5) in Ω and the fate of the disease are determine and control by the basic reproduction number.

Model Analysis

2.1. Positivity of solution of model.

Theorem 2.1. *Suppose $S(0) \geq 0, I(0) \geq 0, R(0) \geq 0$, the solutions $(S(t), I(t), R(t))$ of model equation (5) are positively invariant for all $t > 0$.*

Proof. Let $\Omega = \sup\{t > 0 \mid S > 0, I > 0, R > 0\}$ for the first equation, $\frac{dS}{dt} = b - dS - f(S,I)S + \gamma R + \varpi I - u_1 S$, where $f(S,I) = \frac{kSI}{(1+\alpha S+\beta I)}$. The integrating factor(I.F.) is, $\exp\int_0^\Omega (f(I,S)+d+u_1)\,dt$. We multiply the inequality above by the integrating factor and we obtain $\frac{d}{dt}S(t)\exp\int_0^\Omega (f(I,S)+d+u_1)\,dt \geq (b + \gamma R + \varpi I)\exp\int (f(I,S) + d + u_1)\,dt$. We solve the inequality and obtain $S(t)\exp\{(f(I,S) + d + u_1)t\} - S(0) \geq$
$int_0^\Omega (b + \gamma R + \varpi I)\exp\{\int_0^\Omega (f(I,S) + d + u_1)m\}dm$. Therefore, $S(t)$ becomes
$S(t) \geq S(0)\exp\{-(f(I,S) + d + u_1)t\} + \exp\{-(f(I,S) + d + u_1)t\} \times \int_0^\Omega (b + \gamma R + \varpi I)\exp\{\int_0^\Omega (f(I,S) + d + u_1)m\}dm > 0$. This can as well be shown for $I(t) > 0, R(t) > 0$ respectively.

2.2. Boundedness of the solution of model.

Theorem 2.2. *All solutions (S, I, R) of model (5) are bounded.*

Proof. From the model equation(5), we obtain $\frac{dN(t)}{dt} = \frac{dN}{dt} = b - dN - \rho I$. For the proof, it should be noted that $0 \leq I(t) \leq N$. Adding the first three equations of model (5) we obtained: $\frac{dN(t)}{dt} = b - dN - \rho I\ b - (d+\rho)N(t) \leq \frac{dN(t)}{dt} \leq b - dN(t)$
Hence respectively, $\frac{b}{d+\rho} \leq \liminf_{t\to\infty} N(t) \leq \limsup_{t\to\infty} N(t) \leq \frac{b}{d}$, as required. Therefore, all solutions of the model (5) are bounded. The feasible region for the total population is
$\Delta = \{(S, I, R) \in \mathbb{R}^3_+, N(t) \leq \frac{b}{d}\}$ The Δ is defined as the positively invariant region with respect to the model equation (5), therefore the model equation is mathematically acceptable and epidemiologically significant in Δ.

2.3. **The invariant Region.** The non-linear SIRS model(5) will be analyzed in a biologically feasible region as follows: The system (5)is with total population size $N = S + I + R$. We consider the feasible region:$\Delta \subset \mathbb{R}^3_+$ with $\Delta = \{(S, I, R) \in \mathbb{R}^3_+, N(t) \leq \frac{b}{d}\}$.
The following methods are explore to establish the positive invariance Δ(i.e, solutions in Ω remains in Δ for all $t > 0$).The rate of change of the total population under consideration is given below; $\frac{dN(t)}{dt} = b - dN - \rho I$.
A standard comparison theorem [48] can then be used to verify that $N \leq Ne^{-dt} + \frac{b}{d}(1 - e^{-dt})$. More importantly, $N(t) \leq \frac{b}{d}$, If $N(0) \leq \frac{b}{d}$, respectively. Thus, the region Ω is positively invariant. Hence, it is sufficient to consider the dynamics of the flow generated by (5) in Ω. In this region, the model can be considered as been epidemiologically and mathematically well posed. Thus,every solution of the model equation(5) with initial conditions in Ω remains in Ω for all $t > 0$. Therefore, the w-limit sets of the system (5) are in Ω. We hereby summarized this result below.

Theorem 2.3. *The region $\Delta \subset \mathbb{R}^3_+$ is positively invariant for the model(5) with multiple constant control interventions(vaccination and treatments) and non-negative initial conditions in $\mathbb{R}^3_+$.*

2.4. **The Basic Reproduction number.** The basic reproduction number, R_0, is the average number of secondary infectious generated when one infected person is introduced into a host population where everyone is susceptible. The basic reproduction number is a threshold quantity thats helps to determine whether an outbreak of infectious disease dies out or spreads in a community.When $R_0 \leq 1$, the disease vanishes without the need for any medical strategies. On another case when $R_0 > 1$, the disease becomes endemic and requires some forms of control strategies to come into place. Here, se consider the existence of the equilbrium solution of the system (5) by setting each of the equation to zero. We obtained the disease-free Equilibrium at $D_0 = [S^* = \frac{b(d+\gamma)}{d(d+\gamma+u_1)}, I^* = 0, R^* = \frac{u_1 b}{d(d+\gamma+u_1)}]$.
What are the conditions for an epidemic to occur? An epidemic occurs if the number of infected persons increases i.e. $\frac{dI}{dt} > 0$ such that
$\frac{kSI}{(1+\alpha S+\beta I)} - (d+\varpi+\mu+\rho)I - u_2 I > 0$, since $S^* = \frac{b(d+\gamma)}{d(d+\gamma+u_1)}$,$I^* = 0$ at equilibrium.
At the beginning of an epidemic, almost everyone (except the index case) is susceptible. So we can say that
$S = \frac{b(d+\gamma)}{d(d+\gamma+u_1)}$,by substituting $S = \frac{b(d+\gamma)}{d(d+\gamma+u_1)}$ into $\frac{dI}{dt} > 0$, we obtain the inequality
$\frac{kb(d+\gamma)}{d(d+\gamma+u_1)(d+\varpi+\mu+\rho+u_2)+(\alpha b(d+\gamma))(d+\varpi+\mu+\rho+u_2)} > 1$
Hence, the reproduction number (R_0) without constant control is given by

$$R_0 = \frac{kb(d+\gamma)}{d(d+\gamma)(d+\varpi+\mu+\rho)+(\alpha b(d+\gamma))(d+\varpi+\mu+\rho)}.$$

Hence, we obtained the effective reproduction number (R_e) in the presence of the two constant control is given by

(7)
$$R_e = \frac{kb(d+\gamma)}{d(d+\gamma+u_1)(d+\varpi+\mu+\rho+u_2)+(\alpha b(d+\gamma))(d+\varpi+\mu+\rho+u_2)}.$$

2.5. **Further Analysis of the model.** We define $s = \frac{S}{N}$, $i = \frac{I}{N}$, $r = \frac{R}{N}$ as the proportion for the classes S, I, R respectively. Then by differentiating with respect to time t, it is easy to verify that s, i, and r satisfy the system of differential equation. $\frac{dS}{dt} = s\frac{dN}{dt} + N\frac{ds}{dt}$, $\frac{dI}{dt} = i\frac{dN}{dt} + N\frac{di}{dt}$, $\frac{dR}{dt} = r\frac{dN}{dt} + N\frac{dr}{dt}$ Hence the system(5) becomes;

$$\begin{array}{rcl} \frac{ds}{dt} & = & \frac{bs}{N} - \rho is - \frac{b}{N} + \frac{ksi}{1+(\alpha s+\beta i)N} - \gamma r - \varpi i + u_1 s, \\ \frac{di}{dt} & = & \frac{bi}{N} - \rho i^2 - \frac{ksi}{1+(\alpha s+\beta i)N} + (\varpi + \mu + \rho + u_2 i), \\ \frac{dr}{dt} & = & \frac{br}{N} - \rho ir - \mu r + \gamma r - u_1 s - u_2 i. \end{array} \tag{8}$$

with $s + i + r = 1$

It is observed that system (8) involves the total population size N. We now reduce the system(8) to a three dimensional system by eliminating r since $r = 1 - s - i$ in the feasible region Ω(i.e, where the model makes some epidemiological and biological sense)

$$\begin{array}{rcl} \frac{ds}{dt} & = & \frac{bs}{N} - \rho is - \frac{b}{N} + \frac{ksi}{1+(\alpha s+\beta i)N} - \gamma + \gamma s + \gamma i - \varpi i + u_1 s, \\ \frac{di}{dt} & = & \frac{bi}{N} - \rho i^2 - \frac{ksi}{1+(\alpha s+\beta i)N} + (\varpi + \mu + \rho + u_2)i, \\ \frac{dN}{dt} & = & (\frac{b}{N} - d - \rho i)N. \end{array} \tag{9}$$

where $\Omega = \{(s, i, N) \in \mathbb{R}^3_+ : 0 \leq s, 0 \leq i, s + i \leq 1, N \leq \frac{b}{d}\}$

From the system (9) we observed that the first and second equation depend on the total population size N, So substituting for $\frac{b}{N} = d + \rho i$ into the first and second equations of the system, we obtain the following system:

$$\begin{array}{rcl} \frac{ds}{dt} & = & ds - d - \rho i + \frac{ksi}{1+(\alpha s+\beta i)d} - \gamma + \gamma s + \gamma i - \varpi i + u_1 s, \\ \frac{di}{dt} & = & i(d + \rho i) - \rho i^2 - \frac{ksi}{1+(\alpha s+\beta i)d} + (\varpi + \mu + \rho + u_2)i. \end{array} \tag{10}$$

For convenience, we still use $s = S$, $i = I$ and $r = R$. The system becomes

$$\begin{array}{rcl} \frac{dS}{dt} & = & dS - d - \rho I + \frac{kSI}{1+(\alpha S+\beta I)d} - \gamma + \gamma S + \gamma I - \varpi I + u_1 S, \\ \frac{dI}{dt} & = & I(d + \rho I) - \rho I^2 - \frac{kSI}{1+(\alpha S+\beta I)d} + (\varpi + \mu + \rho + u_2)I. \end{array} \tag{11}$$

It can be verified that the region $\Gamma = \{(S, I) \in \mathbb{R}^2_+ : 0 \leq S, 0 \leq I, S + I \leq 1\}$ is positively invariant with respect to system (11), where $\mathbb{R}^2_+$ including its lower dimensional faces. We denote the boundary and the interior of Γ by $\delta\Gamma$ and $\dot{\Gamma}$ respectively. Thus, the solution of the system (11) is bounded.

3. Mathematical Analysis of the Optimal Control Techniques applied to the model under consideration

The control interventions $0 \leq u_1(t) \leq 1$ and $0 \leq u_2(t) \leq 1$ are the use of vaccination and treatment. Here, we replace the constant control interventions (vaccination and treatment) used in (5) with time dependent control interventions (vaccination and treatment)in our model equation which is given by

$$\begin{array}{rcl} \frac{dS}{dt} & = & dS - d - \rho I + \frac{kSI}{1+(\alpha S+\beta I)d} - \gamma + \gamma S + \gamma I - \varpi I + u_1(t)S, \\ \frac{dI}{dt} & = & dI - \frac{kSI}{1+(\alpha S+\beta I)d} + (\varpi + \mu + \rho + u_2(t))I. \end{array} \tag{12}$$

with initial conditions
$S(0) \geq 0$, $I(0) \geq 0$
The Objective functional is defined as follows

$$J(u_1(t), u_2(t)) = \int_0^T (p_1 I + p_2 u_1(t)^2 + p_3 u_2(t)^2)\, dt. \tag{13}$$

Our goal is to minimize the total number of infective individuals and the cost associated with the use of vaccination $u_1(t)$ and treatment $u_2(t)$ on $[0, T]$. Hence, we seek an optimal control u_1^*, u_2^*

$$J(u_1^*, u_2^*) = \min_{u_1, u_2} \{J(u_1, u_2) | u_1, u_2 \in U\}, \tag{14}$$

where the control set U is defined as

$$U = \{(u_1(t), u_2(t) | u_i : [0, T] \to [0, 1], \textit{Lebesgue measurable } i = 1, 2)\} \tag{15}$$

where $u_1(t)$, $u_2(t)$ are measurable functions such that: $0 \leq u_1(t) \leq 1$, $0 \leq u_2(t) \leq 1$ $t \in [0, T]$. p_1, p_2, p_3 are weight parameters which describe the comparative importance of the terms in the functional. We consider a quadratic cost on the control, which is the simplest and widest used nonlinear representation of vaccination cost.The quadratic term is particularly chosen to describe the nonlinear behaviour of the cost of implementing the vaccination and treatment.

3.1. **Derivation of the Optimal Control Problem.** We intend to find $0 \leq u_1(t) \leq 1$, $0 \leq u_2(t) \leq 1$ for $t \in [0, T]$, to minimize

$$J(u_1(t), u_2(t)) = \int_0^T (p_1 I + p_2 u_1(t)^2 + p_3 u_2(t)^2)\, dt \tag{16}$$

subject to the system of equation (12)

$$\begin{array}{rcl} \frac{dS}{dt} & = & dS - d - \rho I + \frac{kSI}{1+(\alpha S+\beta I)d} - \gamma + \gamma S + \gamma I - \varpi I + u_1(t)S, \\ \frac{dI}{dt} & = & dI - \frac{kSI}{1+(\alpha S+\beta I)d} + (\varpi + \mu + \rho + u_2(t))I. \end{array} \tag{17}$$

and

$$S(0) \geq 0\ , I(0) \geq 0 \tag{18}$$

where $u_1(t)$, $u_2(t)$ are measurable function such that the control constraint

$$U = \{(u_1(t), u_2(t))|0 \leq u_1(t) \leq 1, 0 \leq u_2(t) \leq 1, t \in [0, T]\}.$$

Our goal is to minimize the total number of individuals who are infected while at the same time also minimizing the cost of treatment and the cost of vaccination of the population given the initial population sizes of all two classes $S(0)$, $I(0)$. The first term in the objective functional, $p_1 I$ stands for the total number of individuals who are infected and is taken as a measure of the death associated with the outbreak. The term $p_2 u_1(t)^2$ represent the cost of vaccination while the term $p_3 u_2(t)^2$ represent the cost associated with the treatments. We consider in this paper a quadratic cost on the control because the cost follows a non-linear representation. Hence $u_1(t)$,$u_2(t)$ are Lebesgue integrable and are piecewise continuous and integrable where p_1, p_2, and p_3 are relative weights fixed to the cost of minimizing the total number of infected individual, cost of vaccination and treatment respectively.Hence, we seek an optimal control pair (u_1^*, u_2^*) such that

$$J(u_1^*, u_2^*) = \min\{J(u_1(t), u_2(t)) : (u_1(t), u_2(t)) \in U\}, \tag{19}$$

U is the control set defined by above.

3.2. **Existence and Uniqueness of the control.**

Theorem 3.1. *Suppose the objective functional* $J(u_1(t), u_2(t)) = \min\{J(u_1(t), u_2(t)) = \int_0^T (p_1 I + p_2 u_1(t)^2 + p_3 u_2(t)^2)\, dt$ *where* $u = \{u_1(t), u_2(t) : u_i(t)$ *measurable* $0 \leq u_1(t) \leq 1, 0 \leq u_2(t) \leq 1, t \in [0, T] \in \mathbb{R}^+$ *for* $i = 1, 2...\}$ *subject to the dynamic constraints of system equation* (12) *with* $S(0) = S_0$, $I_(0) = I_0$, *then there exists an optimal control* $u^* = (u_1^*, u_2^*)$ *such that* $\min_{u_1, u_2 \in u} J(u_1(t), u_2(t)) = J(u_1(t)^*, u_2(t)^*)$
Subject to the control system (12) *with the initial conditions.*

Proof. To prove the existence of an optimal control pair we use the result in [43] and [27].The control and the state variables are non-negative values and are non-empty. In the minimization problem, the necessary convexity of the objective functional in $u_1(t)$ and $u_2(t)$ are satisfied. The control variables $u_1(t), u_2(t) \in U$ are also convex and closed by definition. The optimal system is bounded which determines compactness needed for the existence of the optimal control. Furthermore, the integrand in the objective functional which is $(p_2 u_1(t)^2 + p_3 u_2(t)^2)$ is convex on the control set U. There exist constants b_1, $b_2 > 0$ and $\beta > 1$ such that the integrand of the objective functional J is convex and satisfies $J(u_1(t), u_2(t)) \geq b_1(|u_1(t)^2|^2 + |u_2(t)^2|^2)^{\frac{\beta}{2}} - b_2$. By standard control arguments

involving the bounds on the control, we conclude

$$u_1^* = \begin{cases} 0 \; if \; \zeta_1^* \leq 0, \\ \zeta_1^* \; if \; 0 < \zeta_1^* < 1, \\ 1 \; if \; \zeta_1^* \geq 1 \end{cases} \tag{20}$$

where

$$\zeta_1^* = -\frac{(\lambda_S)S}{2p_2}$$

$$u_2^* = \begin{cases} 0 \; if \; \zeta_2^* \leq 0, \\ \zeta_2^* \; if \; 0 < \zeta_2^* < 1, \\ 1 \; if \; \zeta_2^* \geq 1 \end{cases} \tag{21}$$

where

$$\zeta_2^* = -\frac{(\lambda_I)I}{2p_3}$$

By the apriori boundedness of the state system, adjoint system and the resulting Lipschitz structure of the ODEs, we obtain the uniqueness of the optimal control for small T (T means t final).The uniqueness of the optimal control follows from the uniqueness of the optimality system, which consist of (29), (30) and (31) with characterization (32) and (33). We impose a bound on the length of time interval in order to guarantee the uniqueness of the optimality system. The smallness restriction of the length on the state problem has initial values and the adjoint problem has final values. This restriction is very common in control problems[43,49]. Hence, we show the convexity of the control set and the convexity of the functional.

- Convexity of the control set U

We prove the convexity of the control set U by defining the control set $U = [0, 1]$. Suppose for any given two points $a, b \in U$, such that $a = (a_1, a_2)$ and $b = (b_1, b_2)$ Then by the definition of convexity we have that
$\alpha a_i + (1 - \alpha)b_i \in [0, 1]^2$ for all $\alpha \in [0, 1]$, $i \in 1, 2$
Therefore, if $\alpha a + (1 - \alpha)b \in U$ implies that U is convex.

- Convexity of the functional

Here, we prove the convexity of the functional. Suppose $g(S, I, u_1, u_2, t) = p_1 I + p_2 u_1^2 + p_3 u_2^2$
We define the set $V(S, I, u_1, u_2, t) = \{g(S, I, u_1, u_2, t) + \chi_i\}$ for $i = 1, 2$

Then, we have
$\theta(g(S,I,u_1,u_2,t)+\chi_1)+(1-\theta)(g(S,I,u_1,u_2,t)+\chi_2)$
$=\theta(p_1I+p_2u_1(t)^2+p_3u_2(t)^2)+(1-\theta)(p_1I+p_2u_{1,1}(t)^2+p_3u_{2,1}(t)^2+\theta\chi_1+(1-\theta)\chi_2)$
$=p_1I+\theta p_2u_1(t)^2+(1-\theta)p_2u_{1,1}(t)^2+\theta p_3u_2(t)^2+(1-\theta)p_3u_{2,1}(t)^2+\theta\chi_1+(1-\theta)\chi_2$
Thus, if $\theta p_2u_1(t)^2+(1-\theta)p_2u_{1,1}(t)^2\in g(S,I,u_1,u_2,t)$ it implies that the functional J is convex.

3.3. **Necessary conditions of the control.** We use the Pontryagin's Maximum Principle[85] because it is a constrained control problem. Hence, we give the minimized pointwise Hamiltonian as follows. The necessary conditions that an optimal control must satisfy come from Pontryagin's Maximum Principle[84]. This principle converts the system(12) and equation(13) objective functional into a problem minimizing pointwise a Hamiltonian H,with respect to u_1 and u_2.
Theorem 5.: Given an optimal control pair $U^*(t)=(u_1^*,u_2^*)$ and a solution $X^*(t)=(S^*(t),I^*(t))$ of the corresponding equation (12) then there exist adjoint variables $\lambda_S(t),\lambda_I(t)$ which satisfies the following:
$\acute{\lambda_S}=-\frac{\partial H}{\partial S}=-[\lambda_S(d+\frac{kI(1+\beta Id)}{(1+d\alpha S+d\beta I)^2}+\gamma+u_1))$
$+\lambda_I(\frac{kI(1+\beta Id)}{(1+d\alpha S+d\beta I)^2})]$,

$\acute{\lambda_I}=-\frac{\partial H}{\partial I}=-[p_1+\lambda_S(-\rho+\frac{kS(1+\beta Sd)}{(1+d\alpha S+d\beta I)^2}+\gamma-\varpi)$
$+\lambda_I(d-\frac{kS(1+\beta Sd)}{(1+d\alpha S+d\beta I)^2}+(\varpi+\mu+\rho+u_2))]$.
with the final conditions

$$\lambda_S(T)=\lambda_I(T)=0. \tag{22}$$

Furthermore, we find the optimal control u_1^* and u_2^*,

$$u_1^*=\min\{\max(0,-\frac{(\lambda_S)S}{2p_2},1\}, \tag{23}$$

$$u_2^*=\min\{\max(0,-\frac{(\lambda_I)I}{2p_3},1\}. \tag{24}$$

Proof. We form the Hamiltonian H given by
$H(S,I)=(p_1I_h+p_2u_1^2+p_3u_2^2)+\lambda_S(dS-d-\rho I+\frac{kSI}{1+(\alpha S+\beta I)d}-\gamma+\gamma S+\gamma I-$
$\varpi I+u_1S)+\lambda_I(I(d+\rho I)-\rho I^2-\frac{kSI}{1+(\alpha S+\beta I)d}+(\varpi+\mu+\rho+u_2)I)$.
and the transversality conditions

$$\lambda_S(T)=\lambda_I(T)=0 \tag{25}$$

Therefore, we differentiate the Hamiltonian with respect to u_1 and u_2 in the interior of u_1 and u_2 we obtain the optimality condition that follows:

$$\begin{array}{rcl} \frac{\partial H}{du_1} & = & 2p_2u_1 + \lambda_S S(t) = 0, \\ \frac{\partial H}{du_2} & = & 2p_3u_2 - \lambda_I I(t) = 0. \end{array} \tag{26}$$

From these equations, we obtain the optimal control pair (u_1^*, u_2^*) as stated below:

$$u_1^* = -\frac{(\lambda_S)S}{2p_2}, \tag{27}$$

$$u_2^* = -\frac{(\lambda_I)I}{2p_3}. \tag{28}$$

We impose some bounds on the control variables: $0 \leq u_1 \leq 1$ and $0 \leq u_2 \leq 1$ to yield(23) and (24) as required. Therefore our resulting optimality system is given in the next section.

3.4. **Optimality System.** Therefore, our resulting optimality system is given by:
State equations:

$$\begin{array}{rcl} \frac{dS}{dt} & = & dS - d - \rho I + \frac{kSI}{1+(\alpha S+\beta I)d} - \gamma + \gamma S + \gamma I - \varpi I + u_1(t)S, \\ \frac{dI}{dt} & = & dI - \frac{kSI}{1+(\alpha S+\beta I)d} + (\varpi + \mu + \rho + u_2(t))I. \end{array} \tag{29}$$

and

$$S(0) \geq 0 \ , I(0) \geq 0 \tag{30}$$

Adjoint equations: $\lambda_S^{'} = -\frac{\partial H}{\partial S} = -[\lambda_S(d + \frac{kI(1+\beta Id)}{(1+d\alpha S+d\beta I)^2} + \gamma + u_1))$
$+ \lambda_I(\frac{kI(1+\beta Id)}{(1+d\alpha S+d\beta I)^2})]$,

$\lambda_I^{'} = -\frac{\partial H}{\partial I} = -[p_1 + \lambda_S(-\rho + \frac{kS(1+\beta Sd)}{(1+d\alpha S+d\beta I)^2} + \gamma - \varpi)$
$+ \lambda_I(d - \frac{kS(1+\beta Sd)}{(1+d\alpha S+d\beta I)^2} + (\varpi + \mu + \rho + u_2))]$.
Transversality equations:

$$\lambda_S(T) = \lambda_I(T) = 0. \tag{31}$$

Characterization of the optimal control u_1^* and u_2^*:
$\frac{\partial H}{\partial u_1} = \frac{\partial H}{\partial u_2} = 0$ at $u_1 = u_1^*$,$u_2 = u_2^*$ on the set $\{t \in [0,T] : 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1\}$.
That is:

$$u_1^* = \begin{cases} 0 \ if \ \zeta_1^* \leq 0, \\ \zeta_1^* \ if \ 0 < \zeta_1^* < 1, \\ 1 \ if \ \zeta_1^* \geq 1. \end{cases} \tag{32}$$

where

$$\zeta_1^* = -\frac{(\lambda_S)S}{2p_2}$$

$$u_2^* = \begin{cases} 0 \; if \; \zeta_2^* \leq 0, \\ \zeta_2^* \; if \; 0 < \zeta_2^* < 1, \\ 1 \; if \; \zeta_2^* \geq 1. \end{cases} \tag{33}$$

where

$$\zeta_2^* = -\frac{(\lambda_I)I}{2p_3}.$$

4. Numerical Simulations Results and Discussions

Here, we study the numerical approximation of optimal transmission parameter control for a SIRS epidemic control model with the asymptotic transmission. We obtained the optimal control by calculating the optimality system which consists of the state system, the adjoint, transversality equations and characterization of optimal control.We use the iterative scheme to calculate the optimality system. We investigate a deterministic model with non-linear transmission function and study the impact of vaccination and treatment on the SIRS epidemic model.The numerical algorithm presented below is a forward-backward sweep method. We begin by solving the state equations with a guess for the controls over the simulated time using a forward fourth order Runge-Kutta scheme. Hence, the co-state equations are simultaneously solved using a backward fourth order Runge-Kutta scheme with the transversality conditions. Then, the controls are updated by using a convex combination of the previous controls and the value from the characterizations of u_1^* and u_2^*. This process is repeated and iteration is stopped if the values of unknowns at the previous iteration are very close to the ones at the present iteration[47]. The simulation which were carried out by using the following values: $d = 0.02$, $\rho = 0.1$, $\mu = 0.005$, $d = 0.75$,$v = 0.001$ $\gamma = 0.00137$, $S(0) = 0.55$, $I(0) = 0.05$,$k = 0.9$,$\alpha = 0.31$,$\beta = 0.47$ $C_1 = 10$, $C_2 = 1$, $C_3 = 2$
$u_1^* = -\frac{(\lambda_S)S}{2p_2}$,
$u_2^* = -\frac{(\lambda_I)I}{2p_3}$. Considering $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 1$, u_1^* and u_2^* are given by $u_1^* = -\frac{(\lambda_S)S}{2p_2}$ and $u_2^* = -\frac{(\lambda_I)I}{2p_3}$.

Table 2: Table showing numerical values of parameters used in the simulations

Parameter	**Symbol**	**Value**	**Source**
recruitment rate	b	0.03	[71]
disease induced death rate	ρ	0.1	[71]
disease transmission rate	k	0.75	[71]
rate of loss of immunity	μ	0.00137	[assumed]
rate of infected becoming susceptible	ϖ	0.001	[assumed]
natural death rate	d	0.02	[71]
effect of Psychological	α	0.31	[assumed]
effect of sociological	β	0.02	[71]
natural recovery rate	γ	0.005	[assumed]

4.1. **Vaccination strategies.** In this simulation, we set the treatment strategy u_2 only to zero while we activate the use of vaccination control strategy u_1 only. From Figure 2(a), we see that to eliminate the disease in 20days the mass vaccination should be held intensively during the 20days. Using the optimal control u_1 as in Figure 2(a), the dynamics of infected individuals and susceptible individuals could be seen in Figure 1(a) and 1(b) and 3(a). The proportion of infected and susceptible individuals decreases to the minimum due of the activation of vaccination control strategy u_1 while the proportion of susceptible individuals increase without vaccination, the proportion of the infected still decreases without vaccination due constant rate of recovery of some infected individuals who became susceptible again(v).

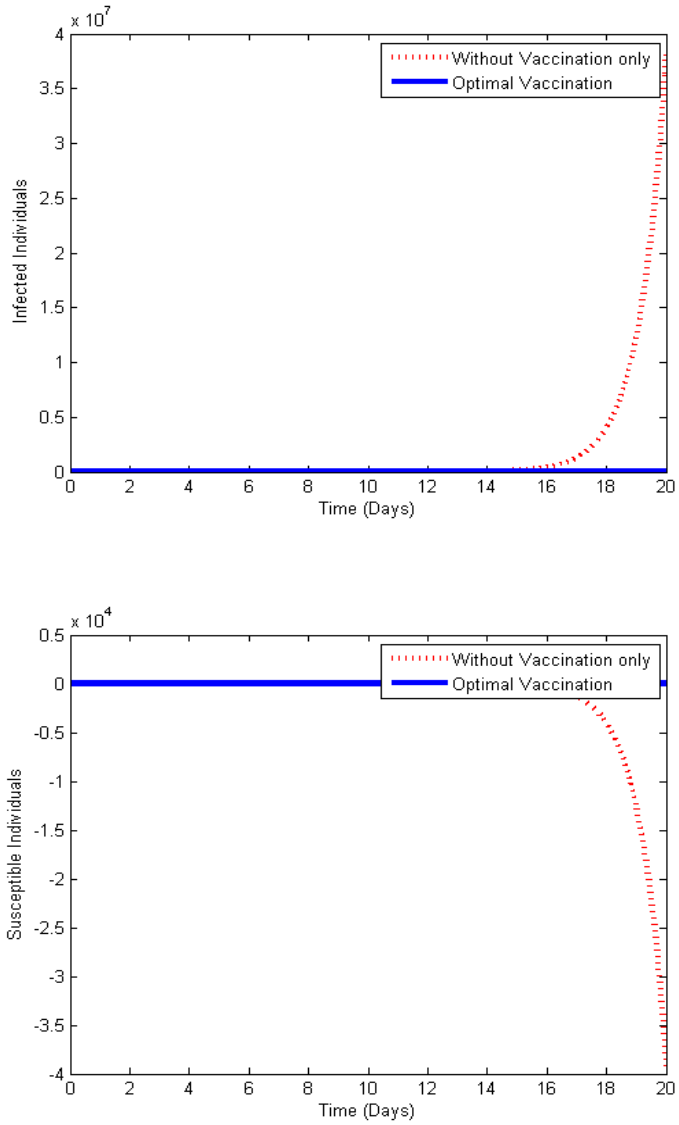


Figure 1: Simulation showing the Optimal solutions for Infected Individual(I) and Susceptible Individual(S) via vaccination only

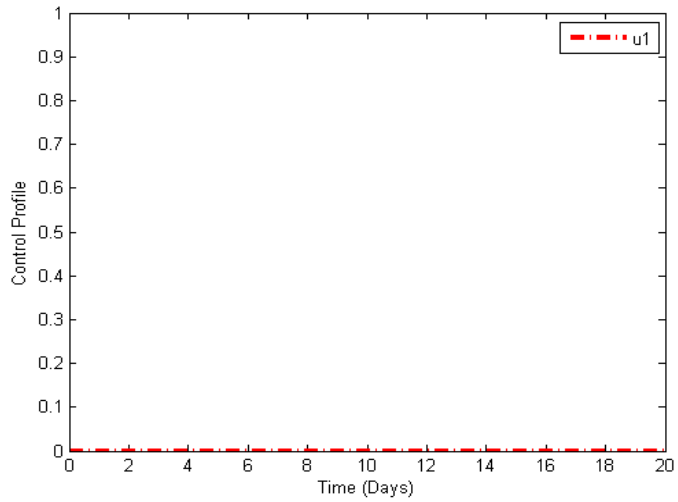


Figure 2: Simulation showing the profile for the Optimal control u_1

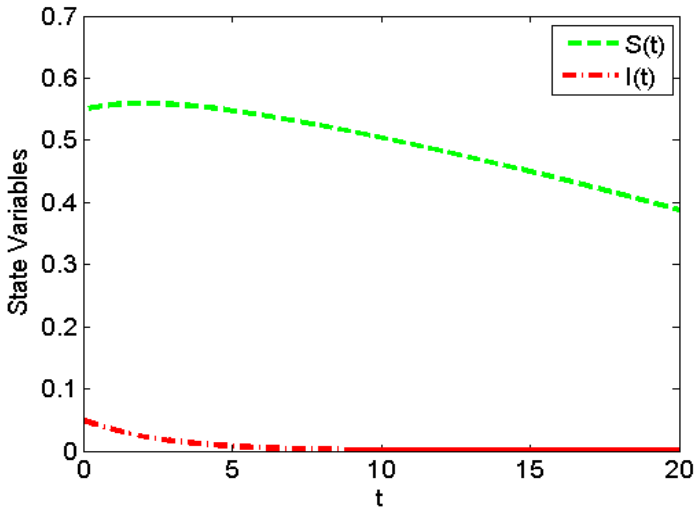
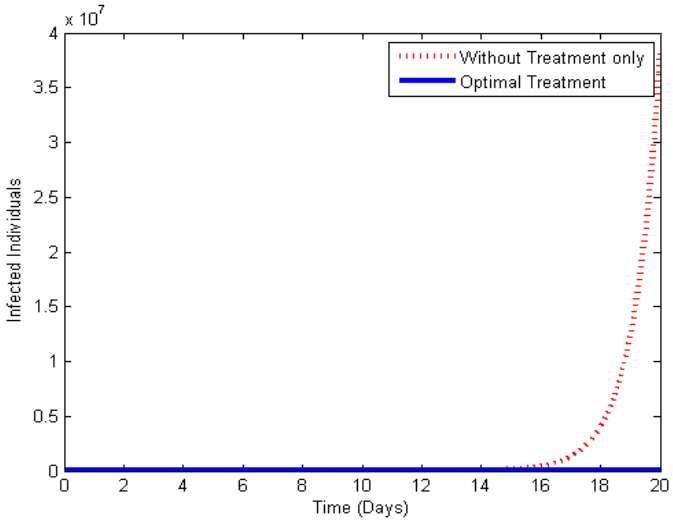


Figure 3: Simulation showing the profile for the Optimal states S and I via vaccination only

4.2. **Treatment strategies.** In this plot, we used the control u_2 on treatment only while the control u_1 on vaccination is set to zero. The profile of the optimal control u_2 in Figure 5(a) was presented. To reduce the disease in 20days, the treatment must be held consistently and intensively almost every day during the 20days. Using the optimal control u_2 as in Figure 5(a), the dynamics of infected individuals and susceptible individuals could be seen in Figure 4(a) and 4(b) and Figure 6(a). The proportion of individuals decreases due to the treatment, while the proportion of infected individuals increases without treatment the susceptible individuals still decreases without treatment.



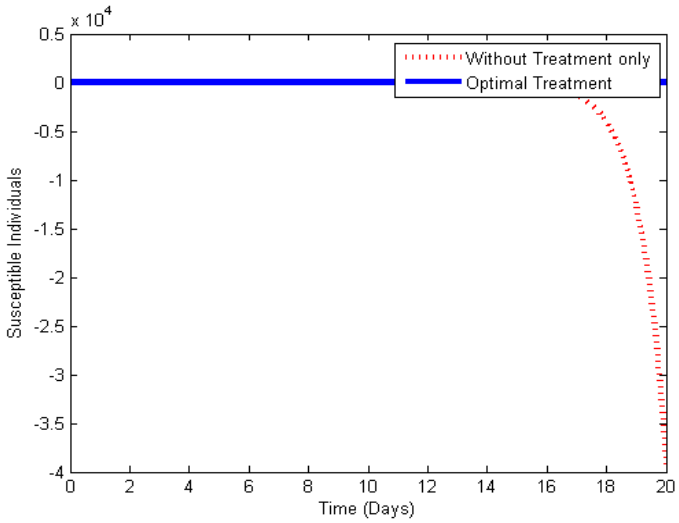


Figure 4: Simulation showing the Optimal solutions for Infected Individual(I) and Susceptible Individual(S) via treatment only

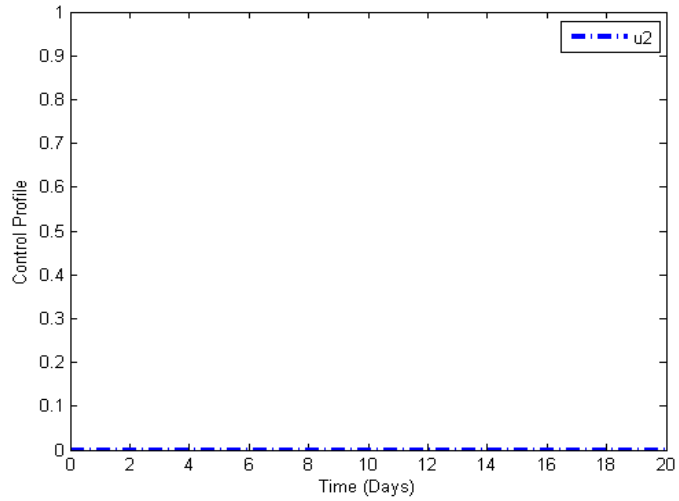


Figure 5: Simulation showing the profile for the Optimal control u_2

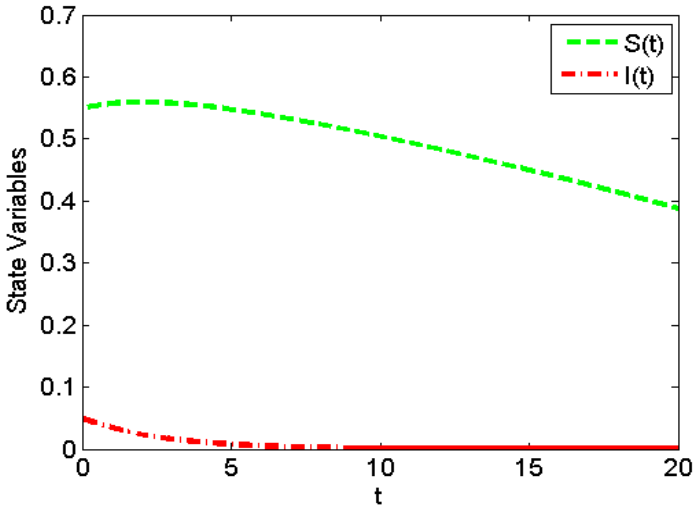


Figure 6: Simulation showing the profile for the Optimal states S and I via treatment only

4.3. **Optimal vaccination and treatment strategies.** In this simulation, the vaccination u_1 and treatment control u_2 are both applied to optimize the cost functional J. The profile of optimal controls u_1 and u_2 can be seen in Figure 8(a).From Figure 8(a), we observe that to eliminate the disease in 20 days, vaccination and treatment must be held intensively. Applying the optimal control u_1 and u_2 in Figure 8(a), the dynamics of infected individuals and susceptible individuals in Figure 9(a) and (b). We observe that due to the control strategies applied, the proportion of infected individuals and susceptible individuals decreases, while the proportion of susceptible and infected individuals increase without control applied.This simulation reveals that the optimal combination of vaccination and treatment strategies can effectively decrease the proportion of infected and susceptible individuals.

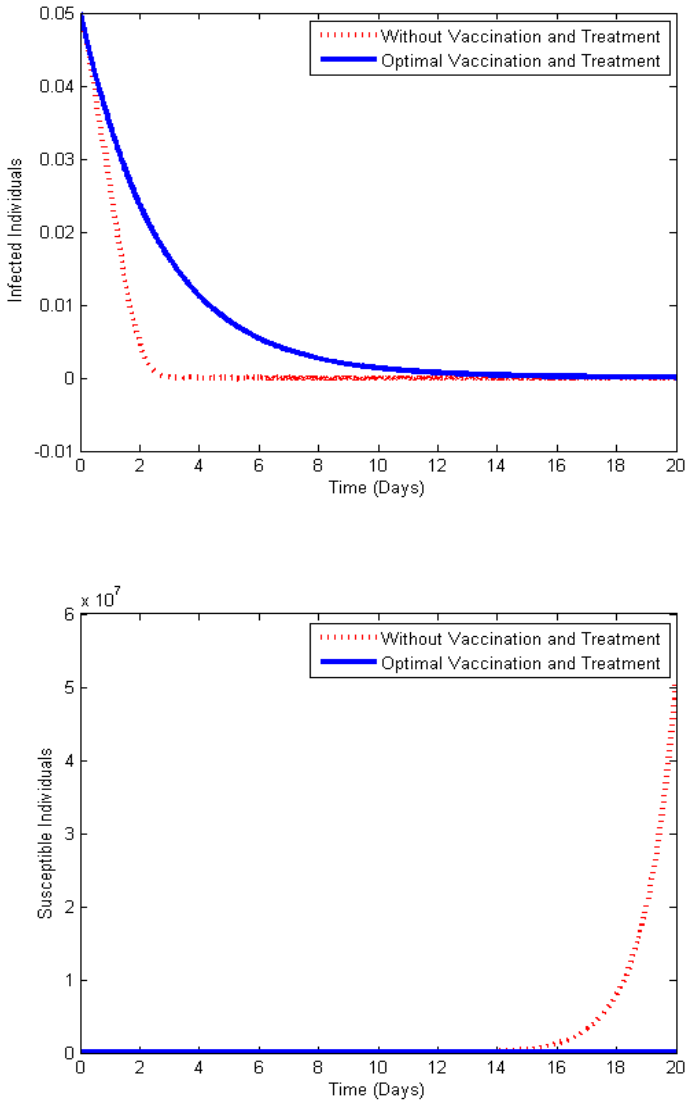


Figure 7: Simulation showing the Optimal solutions for Infected Individual(I) and Susceptible Individual(S) via vaccination and treatment

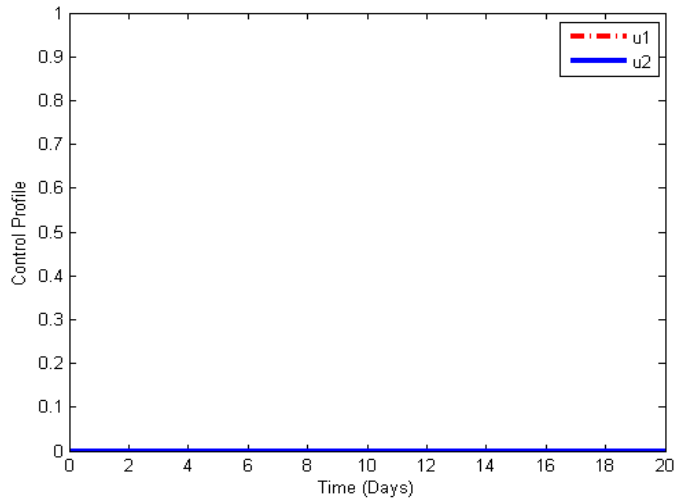


Figure 8: Simulation showing the profile for the Optimal control u_1 and u_2

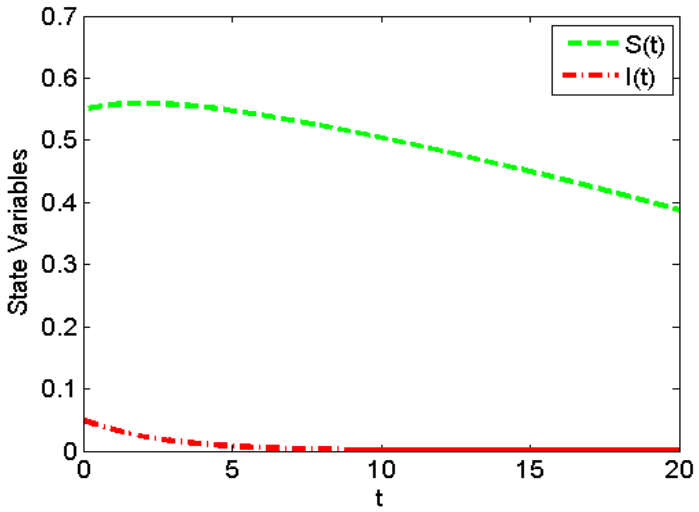


Figure 9: Simulation showing the profile for the Optimal states S and I

Conclusion

In this paper, a compartmental deterministic model is formulated and analyzed for a SIRS epidemic model with the asymptotic transmission, and vaccination and treatment strategies. We obtained the basic reproduction number, R_0, with the two control interventions.The reproduction number, which helps us to determine the threshold level of vaccination required to eradicate any disease. The application of optimal control help us to formulate and analyze the conditions for optimal control of the disease with effective vaccination and treatment interventions.The numerical results showed that prevention by vaccination and the cost put into treatment have a strong impact on the reducing the morbidity and mortality. Therefore, it is concluded that adequate control measures which adhere to these control strategies (vaccination and treatment) would be a very effective way for combating and reducing the morbidity and mortality of any disease in question.This work can be further extended by introducing stochasticity and seasonality into the incidence function.The analytical pattern used in this work can as well be applied and extended to other epidemics models such as those used in the modelling of influenza, HIV/AIDS, Tuberculosis(TB), Malaria, Dengue fever, Zika and Cholera and many others.

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