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On the Existence of Fixed Points Satisfying Graphic Contraction Conditions on an Open Ball

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Abstract

This paper investigates the existence of fixed points in complete metric spaces through the application of Ćirić-type fixed point principles for (α, ψ)-contractive mappings. By formulating the problem within an open ball of a complete metric space, we establish sufficient conditions that guarantee the existence of fixed points under this generalized contractive framework. An illustrative example and corollaries are presented to demonstrate the validity and applicability of the main results.

1. Introduction and Preliminaries

A point $x^* \in X$ is called a fixed point of T if $T(x^*) = x^*$. That is , after applying the mapping T to x^*, the result is still the same point. In a metric space, an open ball is the set of all points whose distance from a given center point is strictly less than a specified radius. The open ball centered at x_0 with radius r is defined as $B(x_0, r) = x \in X : d(x, x_0) < r$. Fixed point results for graphic

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contractions in an open ball state that a function has a fixed point within open ball under specific contractive conditions, such as being a graphic contraction. It provides existence and uniqueness guarantees for a fixed point that satisfies the contraction property in open ball. Results in this area are often presented as extensions and generalizations of Banach contraction principle, by applying the conditions to a smaller domain. Results often require the space to be complete and the contraction to satisfy specific conditions related to the graph, such as being an α-admissible contraction, which provides a generalized framework for finding fixed points under these restricted conditions.

In metric fixed point theory, the contractive conditions functions play an important role for finding solutions of fixed point problems. Due to its importance and simplicity, several authors have it in different directions.

[15] proved some fixed point results in dislocated quasi and quasi $b-$metric spaces. They also proved a theorem of integral type for some rational type generalized contractions in the context of dislocated metric spaces.

[5] investigated fixed point results for almost $(\zeta-\theta)-$contractions in the context of quasi-metric spaces. The study focused on a specific class of $(\zeta-\theta)-$contractions, which exhibit a more relaxed form of contractive property than classical contractions. The research also provided sufficient conditions for the convergence of fixed point sequence.

[2] introduced notion of a function weighted $L-R-$complete dislocated quasi-metric space and generalized concept of bi-completeness by introducing the concept of $L-R-$completeness. [14] introduced a concept of $\alpha-\psi-$contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards, [9] refined the notions and obtain various fixed point results. [7] enlarged the concept of $\alpha-$admissible mappings and obtained useful fixed point theorems. Subsequently, [1] introduced pairs of $\alpha-$admissible mappings satisfying new sufficient contractive conditions, and proved fixed point and common fixed point theorems. Lately, [13] modified the concept of $\alpha-\psi-$contractive mappings and established fixed point results. [11]) introduced a new notion of $\alpha-\psi-$contractive mappings and showed that this is a real generalization for some old results. [3] established fixed point results of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space.

[4] introduced a type of contraction called the $\alpha-$Geraghty contraction in metric spaces, establishing fixed point theorems for such mappings, which generalized existing results, particularly for $\alpha-$admissible functions, and included applications to show the practical relevance and improvement over prior work in fixed point theory, extending concept like $\alpha-$admissibility and using tools like Geraghty functions.

[10] proved fixed point results of $\alpha-$admissible mappings with respect to η, for

modified contractive condition in complete metric space. [12] presented some results on the approximation of fixed points. [6] introduced the concept of an $\alpha-$admissible map with respect to η and modified the $\alpha - \psi-$contractive condition for a pair of mappings and established common fixed point results for two, three, and four mappings in a closed ball in complete dislocated metric spaces. Over the years, fixed point theory has been generlized in multi-directions by several mathematicians.
In this research work, existence of fixed points for mappings satisfying graphic contraction conditions using Ciric-type mapping on open ball of a complete metric space would be derived and proved. Also, some results concerning Ciric-type fixed point for (α, μ)-contraction would be considered.

2. PRELIMINARY RESULTS

The following lemma, example and definitions are required to prove our main results.
Lemma 1 [13]: If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.
Definition 1[14]: Let (X, d), be a metric space. A mapping $T : X \to X$ is $(\alpha, \psi)-$contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$.
Definition 2 [14]: Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is $\alpha-$admissible if $x, y \in X$, $\alpha(x, y) \geq 1$
implies that

$$\alpha(Tx, Ty) \geq 1.$$

Example 1: Let $(0, \infty)$ and T an identity mapping on X. Define $\alpha : X \times X \to [0, +\infty)$ by

$$d(x, y) = \begin{cases} e^{\frac{y}{x}}, & if\ x \geq y,\ x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then T is $\alpha-$admissible.
Definition 3 [13]: Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ two functions. We say that T is $\alpha-$admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$
implies that

$$\alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

If $\eta(x, y) = 1$, then above definition reduces to Definition 2. If $\alpha(x, y) = 1$, then T is called an $\eta-$subadmissible mapping.

Definition 4 [13]: Let $T : X \to X$ and $\alpha_0 : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \alpha(x,y) \geq \eta(x,y) \\ 0 & \text{otherwise} \end{cases}$$

We say that T is α_0-admissible. If $\alpha(x,y) \geq 1$ then $\alpha(x,y) \geq \eta(x,y)$ and so $\alpha(Tx,Ty) \geq \eta(Tx,Ty)$.
This implies that

$$\alpha(Tx,Ty) = 1.$$

Also

$$\alpha_0(x_0,Tx_0) = 1.$$

Moreso, consistent with [8], let (X,d) be a metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $D(G)$ of its vertices coincides with X, and the set $S(G)$ of its edges contains all loops, i.e., $S(G) \supset \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(D(G), S(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length $m (m \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{m}$ of $m+1$ vertices such that $x_0 = x, x_m = y$ and $(x_{n-1}, x_n) \in S(G)$ for $i = 1, ..., m$. A graph G is connected if there is a path between any two vertices. G is weakly connected if G is connected. Now we extend concept of G-contraction as follows:
Definition 5 [8]: defined a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, i.e.,

$$\forall\ x, y \in X, (x,y) \in D(G) \Rightarrow (Tx,Ty) \in S(G)$$

and T decreases weights of edges of G in the following way:

$$\exists\ k \in (0,1),\ \forall\ x, y \in X, (x,y) \in D(G)$$
$$\Rightarrow d(Tx,Ty) \leq kd(x,y).$$

Definition 6: Let (K, d_q), be a metric space endowed with a graph G and $F : K \to K$ be self-mappings. Assume that $r > 0$, $x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, following conditions hold,

$$\forall\ x, y \in B_{d_q}(x_0, r), (x,y) \in S(G) \Rightarrow (Fx, Fy) \in S(G)$$
$$\forall\ x, y \in B_{d_q}(x_0, r), (x,y) \in S(G) \Rightarrow d_q(Fx, Fy) \leq \mu(N_q(x,y)).$$

where

$$M_q(x,y) = max \left\{ d_q(x,y), \frac{d_q(x,Fx) + d_q(y,Fy)}{2}, \frac{d_q(x,Fy) + d_q(y,Fx)}{2} \right\},$$

Then the mappings F is called Ciric μ-graphic contractive mappings. If $\mu(t) = kt$ for some $k \in [0, 1)$, then we say F is G-contractive mappings.

3. MAIN RESULTS

In the following section, Ciric-type fixed point results for (α, μ)-contractive maps in an open ball of a complete metric space are used to obtain the main results.

Theorem 3.1: Let (K, d_q) be a complete metric space and if $F : K \to K$ is α-admissible mapping with respect to β and $\alpha, \beta \in (0, 1)$. Suppose $r > 0$, $x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$ where $\mu : [0, \infty) \to [0, \infty)$, assume that,

$$\alpha(x, y) \geq \beta(x, y)$$

and

$$(x, y) \in B_{d_q}(x_0, r),$$

$$\Rightarrow d_q(Fx, Fy) \leq \mu(Q_q(x, y)), \tag{1}$$

where

$$Q_q(x, y) = max \left\{ d_q(x, y), d_q(x, Fx), d_q(y, Fy), \frac{d_q(x, Fy) + d_q(y, Fx)}{2} \right\}$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0, Fx_0)) \leq r, \tag{2}$$

for all $j \in \mathbb{N}$.
Suppose that the following assertions hold:
(i) $\alpha(x_0, Fx_0) \geq \beta(x_0, Fx_0)$;
(ii) for any sequence $\{x_n\}$ in $B_{d_q}(x_0, r)$ such that $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{x+1})$ for all $n \in \mathbb{N} \cup \{0\}$
and
$x_n \to u \in B_{d_q}(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \beta(x_n, u)$ for all $n \in \mathbb{N} \cup \{0\}$
Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that $Fx^* = x^*$

Proof:
Let x_1 be in K be such that $x_1 = Fx_0, x_2 = Fx_1, x_3 = Fx_2 \ . \ . \ . \ x_n = Fx_{n-1}$.
By assumption $\alpha(x_0, x_1) \geq \beta(x_0, x_1)$ and F is $\alpha-$admissible mapping with respect to β.

we have,

$$\alpha(Fx_0, Fx_1) \geq \beta(Fx_0, Fx_1)$$

from which

$$\alpha(x_1, x_2) \geq \beta(x_1, x_2)$$

which also implies that

$$\alpha(Fx_1, Fx_2) \geq \beta(Fx_1, Fx_2).$$

and

$$\alpha(x_2, x_3) \geq \beta(x_2, x_3)$$

gives

$$\alpha(Fx_2, Fx_3) \geq \beta(Fx_2, Fx_3)$$

Continuing in this way we obtain

$$\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1}) \; \forall \; n \in \mathbb{N} \cup \{0\}.$$

implies that

$$\alpha(Fx_n, Fx_{n+1}) \geq \beta(Fx_n, Fx_{n+1}) \; \forall \; n \in \mathbb{N} \cup \{0\}.$$

it could be shown that

$$x_n \in B_{d_q}(x_0, r) \; \forall \; n \in \mathbb{N}$$

By using (2)

$$d_q(x_0, Fx_0) \leq \sum_{i=0}^{j} \mu^i(d_q(x_0, Fx_0)) \leq r.$$

It follows that

$$x_1 \in B_{d_q}(x_0, r).$$

Assume $x_2, ..., x_j \in B_{d_q}(x_0, r)$ for some $j \in \mathbb{N}$.
(1) gives

$$\begin{aligned} d_q(x_i, x_{i+1}) &= d_q(Fx_{i-1}, Fx_i) \\ &\leq \mu(Q_q(x_{i-1}, x_i)) \end{aligned}$$

But

$$\begin{aligned} Q_q(x_{i-1}, x_i) &= max\left\{d_q(x_{i-1}, x_i), d_q(x_i, x_{i+1}), \frac{d_q(x_{i-1}, x_{i+1})}{2}\right\} \\ &\leq max\left\{d_q(x_{i-1}, x_i), d_q(x_i, x_{i+1}), \frac{d_q(x_{i-1}, x_i) + d_q(x_i, x_{i+1})}{2}\right\} \end{aligned}$$

Hence,

$$Q_q(x_{i-1}, x_i) \leq max\left\{d_q(x_{i-1}, x_i), d_q(x_i, x_{i+1})\right\}, \tag{3}$$

the case $Q_q(x_{i-1}, x_i) = d_q(x_i, x_{i+1})$ is impossible

$$d_q(x_i, x_{i+1}) \leq \mu(d_q(x_i, x_{i+1})) < d_q(x_i, x_{i+1}).$$

Which is a contradiction. Otherwise, in other case if

$$Q_q(x_{i-1}, x_i) = d_q(x_{i-1}, x_i)$$

$$\begin{aligned} d_q(x_i, x_{i+1}) &\leq \mu(d_q(x_{i-1}, x_i)) \\ &\leq \mu^2(d_q(x_{i-2}, x_{i-1})) \\ &\leq ... \leq \mu^i(d_q(x_0, x_1)) \end{aligned}$$

Thus

$$d_q(x_i, x_{i+1}) \leq \mu^i(d_q(x_0, x_1)) \tag{4}$$

Now

$$\begin{aligned} d_q(x_0, x_{j+1}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, x_3) + ... + d_q(x_{i-1}, x_j) + d_q(x_j, x_{j+1}) \\ &\leq \sum_{i=0}^{j} \mu^i(d_q(x_0, x_1)) \\ &\leq r \end{aligned}$$

Thus

$$x_{j+1} \in B_{d_q}(x_0, r).$$

Hence $x_n \in B_{d_q}(x_0, r)$ for all $n \in \mathbb{N}$.
Now equation (4) can be written as

$$d_q(x_n, x_{n+1}) \leq \mu^n(d_q(x_0, x_1)), \tag{5}$$

for all $n \in \mathbb{N}$.
Fix $\epsilon > 0$ and $N \in \mathbb{N}$ such that $n \geq \mathbb{N}$ $\mu^n(d_q(x_0, x_1)) < \epsilon$.
Let $m, n \in \mathbb{N}$ with $m > n > \mathbb{N}$.
Then, by the triangle inequality, we have

$$\begin{aligned} d_q(x_n, x_m) &\leq \sum_{k=n}^{m-1} d_q(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \mu^k(d_q(x_0, x_1)) \\ &\leq \sum_{k=n}^{\infty} \mu^k(d_q(x_0, x_1)) < \epsilon. \end{aligned}$$

therefore,

$$d(x_n, x_m) < \epsilon \ \forall \ m > n \geq \mathbb{N}$$

Hence x_n is a Cauchy sequence in $(B_{d_q}(x_0, r))$.
As every open ball in a complete metric space is complete, so there exists $x^* \in B_{d_q}(x_0, r)$ such that $x_n \to x^*$.
Also

$$lim_{n\to\infty} d_q(x_n, x^*) = 0. \tag{6}$$

based on the given assumption
$\alpha(x_n, x^*) \geq \beta(x_n, x^*)$, for all $n \in \mathbb{N} \cup \{0\}$.
Now from (1)

$$d_q(x_{n+1}, Fx^*) \leq \mu(Q_q(x_n, x^*)). \tag{7}$$

where

$$Q_q(x_n, x^*) = max\left\{d_q(x_n, x^*), d_q(x_n, x_{n+1}), d_q(x^*, Fx^*), \frac{d_q(x_n, Fx^*) + d_q(x^*, x_{n+1})}{2}\right\}$$

If $d_q(x^*, Fx^*) \neq 0$, then $Q_q(x_n, x^*) > 0$ for every n.
Thus

$$d_q(x_{n+1}, Fx^*) \leq \mu(Q_q(x_n, x^*)) < Q_q(x_n, x^*). \tag{8}$$

which on taking limits as $n \to \infty$ gives

$$\begin{aligned} d_q(x^*, Fx^*) &= lim_{n\to\infty} d_q(x_{n+1}, Fx^*) \\ &\leq lim_{n\to\infty} Q_q(x_n, x^*) \\ &= d_q(x^*, Fx^*). \end{aligned}$$

Hence $d_q(x^*, Fx^*) = 0$. The result follows.

Example: If $K = [0, \infty)$ with metric on K is defined by $d_q(x, y) = |x - y|$. Let $F : K \to K$ be defined by,

$$Fx = \begin{cases} x^n & if\ x \in [0,1] \\ x - \frac{1}{3} & if\ x \in (1, \infty) \end{cases}$$

Consider $x_0 = 1, r = 3, \mu(t) = \frac{t}{5}$ and

$$\alpha(x, y) = \begin{cases} 1 & if\ x, y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Now $B_{d_q}(x_0, r) = [0, 1]$. then

$$d_q(x_0, Fx_0) = d_q(1, F1) = d_q(1, \frac{1}{3}) = \big|1 - \frac{1}{3}\big| = \frac{2}{3}$$

$$\sum_{i=0}^{n} \mu^n(d_q(x_0, Fx_0)) = \frac{2}{3}\sum_{i=0}^{n} \frac{1}{5^n} \leq \frac{5}{6} < 3$$

Also if $x, y \in (1, \infty)$, then

$$\big|2x - 2y\big| > \big|x - y\big|$$

$$\big|x - \frac{1}{3} - (y - \frac{1}{3})\big| > \mu(\big|x - y\big|)$$

$$d_q(Fx, Fy) > \mu(d_q(x, y))$$

$$d_q(Fx, Fy) > \mu(Q_q(x, y))$$

Then the contractive condition does not hold on K.
Also if, $x, y \in B_{d_q}(x_0, r)$, then

$$\big|\frac{2x}{3} - \frac{2y}{3}\big| \leq \big|x - y\big|$$

$$\frac{1}{3}\big|x - y\big| \leq \mu(\big|x - y\big|)$$

$$d_q(Fx, Fy) \leq \mu(d_q(x, y)) \leq \mu(Q_q(x, y)).$$

where $\mu = \frac{1}{2}$. If $\beta(x, y) = 1$ in the Theorem 3.1, we have the following theorem:

Theorem 3.2: Let (K, d_q) be a complete metric space and if $F : K \to K$ is α-admissible mapping with respect to β. Suppose $r > 0$, $x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, assume that,

$$\begin{aligned} &x, y \in B_{d_q}(x_0, r),\ \alpha(x, y) \geq 1 \\ &\Rightarrow d_q(Fx, Fy) \leq \mu(Q_q(x, y)), \end{aligned} \tag{9}$$

where

$$Q_q(x,y) = max\left\{d_q(x,y), d_q(x,Fx), d_q(y,Fy), \frac{d_q(x,Fy)+d_q(y,Fx)}{2}\right\}.$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0,Fx_0)) \le r, \ \forall\ j \in \mathbb{N}. \tag{10}$$

Suppose that the following assertions hold:
(i) $\alpha(x_0,Fx_0) \ge 1$;
(ii) for any sequence x_n in $B_{d_q}(x_0,r)$ such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to u \in B_{d_q}(x_0,r)$ as $n \to +\infty$ then $\alpha(x_n,u) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$.
Then there exists a point x^* in $B(x_0,r)$ such that

$$Fx^* = x^*.$$

If $\alpha(x,y) = 1$ in the Theorem 3.1, we have the theorem below

Theorem 3.3: Let (K,d_q) be a complete metric space and if $F: K \to K$ is $\beta-$subadmissible mapping. Suppose $r>0$, $x_0 \in B_{d_q}(x_0,r)$ and $\mu \in \Psi$, assume that,

$$\begin{aligned} &x,y \in B_{d_q}(x_0,r),\ \beta(x,y) \le 1 \\ &\Rightarrow d_q(Fx,Fy) \le \mu(Q_q(x,y)), \end{aligned} \tag{11}$$

where

$$Q_q(x,y) = max\left\{d_q(x,y), d_q(x,Fx), d_q(y,Fy), \frac{d_q(x,Fy)+d_q(y,Fx)}{2}\right\}.$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0,Tx_0)) \le r, \tag{12}$$

for all $j \in \mathbb{N}$.
If following assertions hold:
(i) $\beta(x_0,Fx_0) \le 1$;
(ii) for any sequence x_n in $B_{d_q}(x_0,r)$ such that $\beta(x_n,x_{n+1}) \le 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to u \in B_{d_q}(x_0,r)$ as $n \to +\infty$ then $\beta(x_n,u) \le 1$ for all $n \in \mathbb{N}\cup\{0\}$.

Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that

$$Fx^* = x^*.$$

Corollary 1: If (K, d_q) is a complete metric space and $F : K \to K$ is α-admissible mapping with respect to β. Suppose $r > 0$, $x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, assume that,

$$\begin{aligned} & x, y \in B_{d_q}(x_0, r),\ \alpha(x, y) \geq \beta(x, y) \\ & \Rightarrow d_q(Fx, Fy) \leq \mu(M_q(x, y)), \end{aligned} \tag{13}$$

where

$$M_q(x, y) = max\left\{d(x, y), \frac{d_q(x, Fx) + d_q(y, Fy)}{2}, \frac{d_q(x, Fy) + d_q(y, Fx)}{2}\right\}.$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0, Fx_0)) \leq r, \tag{14}$$

for all $j \in \mathbb{N}$.
Suppose that the following assertions hold:
(i) $\alpha(x_0, Fx_0) \geq \beta(x_0, Fx_0)$;
(ii) for any sequence x_n in $B_{d_q}(x_0, r)$ such that $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to u \in B_{d_q}(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \beta(x_n, u)$ for all $n \in \mathbb{N} \cup \{0\}$.
Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that

$$Fx^* = x^*.$$

If $\beta(x, y) = 1$ in the corollary 1, we have the following corollary.

Corollary 2: If (K, d_q) is a complete metric space and $F : K \to K$ is α-admissible mapping. Suppose $r > 0$, $x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, assume that,

$$\begin{aligned} & x, y \in B_{d_q}(x_0, r), \alpha(x, y) \geq 1 \\ & \Rightarrow d_q(Fx, Fy) \leq \mu(M_q(x, y)), \end{aligned} \tag{15}$$

where

$$M_q(x, y) = max\left\{d(x, y), \frac{d_q(x, Fx) + d_q(y, Fy)}{2}, \frac{d_q(x, Fy) + d_q(y, Fx)}{2}\right\}.$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0, Fx_0)) \leq r, \tag{16}$$

for all $j \in \mathbb{N}$.
Suppose that the following assertions hold:
(i) $\alpha(x_0, Fx_0) \geq 1$;
(ii) for any sequence x_n in $B_{d_q}(x_0, r)$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to u \in B_{d_q}(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that

$$Fx^* = x^*.$$

If $M_q(x, y) = \frac{d_q(x,Fx)+d_q(y,Fy)}{2}$ in the corollary 2, we have the following corollary.

Corollary 3: If (K, d_q) is a complete metric space and $F : K \to K$ is α-admissible mapping. Suppose $r > 0, x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, assume that,

$$\begin{aligned} & x, y \in B_{d_q}(x_0, r),\ \alpha(x, y) \geq 1 \\ & \Rightarrow d_q(Fx, Fy) \leq \mu(M_q(x, y)), \end{aligned} \tag{17}$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0, Tx_0)) \leq r, \tag{18}$$

for all $j \in \mathbb{N}$.
Suppose that the following assertions hold:
(i) $\alpha(x_0, Fx_0) \geq 1$;
(ii) for any sequence x_n in $B_{d_q}(x_0, r)$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to u \in B_{d_q}(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that

$$Fx^* = x^*.$$

If $M_q(x, y) = \frac{d_q(x,Ty)+d_q(y,Tx)}{2}$ in the corollary 2, we have the following corollary.

Corollary 4: If (K, d_q) is a complete metric space and $F : K \to K$ is α-admissible mapping. Suppose $r > 0, x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, assume that,

$$x, y \in B_{d_q}(x_0, r), \alpha(x, y) \geq 1 \tag{19}$$

$$\Rightarrow d_q(Fx, Fy) \leq \mu(M_q(x, y)),$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0, Fx_0)) \leq r, \tag{20}$$

for all $j \in \mathbb{N}$.
Suppose that the following assertions hold:
(i) $\alpha(x_0, Fx_0) \geq 1$;
(ii) for any sequence x_n in $B_{d_q}(x_0, r)$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to u \in B_{d_q}(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that

$$Fx^* = x^*.$$

If $M_q(x, y) = d_q(x, y)$, in the corollary 2, we obtain the following corollary.

Corollary 5: If (K, d_q) is a complete metric space and $F : K \to K$ is α-admissible mapping with respect to β. Suppose $r > 0$, $x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, assume that,

$$x, y \in B_{d_q}(x_0, r), \alpha(x, y) \geq \beta(x, y) \tag{21}$$

$$\Rightarrow d_q(Fx, Fy) \leq \mu(d_q(x, y)),$$

and

$$\sum_{i=0}^{j} \psi^i(d_q(x_0, Fx_0)) \leq r, \tag{22}$$

for all $j \in \mathbb{N}$.
Suppose that the following assertions hold:
(i) $\alpha(x_0, Fx_0) \geq \beta(x_0, Fx_0)$;
(ii) for any sequence x_n in $B_{d_q}(x_0, r)$ such that $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to u \in B_{d_q}(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \beta(x_n, x_u)$ for all $n \in \mathbb{N} \cup \{0\}$.
Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that

$$Fx^* = x^*.$$

If $\beta(x, y) = 1, M_q(x, y) = d_q(x, y)$ in the corollary 2, we have the following corollary.

Corollary 6: If (K, d_q) is a complete metric space and $F: K \to K$ is α-admissible mapping. Suppose $r > 0, x_0 \in B_{d_q}(x_0, r)$ and $\mu \in \Psi$, assume that,

$$x, y \in B_{d_q}(x_0, r), \alpha(x, y) \geq 1 \tag{23}$$
$$\Rightarrow d_q(Fx, Fy) \leq \mu(d_q(x, y)),$$

and

$$\sum_{i=0}^{j} \mu^i(d_q(x_0, Fx_0)) \leq r, \tag{24}$$

for all $j \in \mathbb{N}$.
Suppose that the following assertions hold:
(i) $\alpha(x_0, Fx_0) \geq 1$;
(ii) for any sequence x_n in $B_{d_q}(x_0, r)$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to u \in B_{d_q}(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then, there exists a point x^* in $B_{d_q}(x_0, r)$ such that

$$Fx^* = x^*.$$

Theorem 3.4: Let (K, d_q), be a complete metric space endowed with a graph G and $F: K \to K$ be ciric μ-graphic contractive mappings and $x_0 \in B_{d_q}(x_0, r)$. Suppose that the following assertions hold:
(i) $(x_0, Fx_0) \in E(G)$ and $\sum_{i=0}^{j} \mu^i(d_q(x_0, Fx_0)) < r$ for all $j \in \mathbb{N}$;
(ii) if $\{x_n\}$ is a sequence in $B_{d_q}(x_0, r)$ such that $(x_n, x_{n+1}) \in S(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in S(G)$ for all $n \in \mathbb{N}$.
Then F has a fixed point.
Proof: Define, $\alpha: K^2 \to (-\infty, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & if\ (x, y) \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

First we prove that the mapping F is α-admissible. Let $x, y \in B_{d_q}(x_0, r)$ with $\alpha(x, y) \geq 1$, then $(x, y) \in S(G)$. As F is ciric μ-graphic contractive mappings, $(Fx, Fy) \in S(G)$.
That is, $\alpha(Fx, Fy) \geq 1$. Thus F is α-admissible mapping.
From (2) there exists x_0 such that $(x_0, Fx_0) \in E(G)$.
That is, $\alpha(x_0, Fx_0) \geq 1$. If $x, y \in B(x_0, r)$ with $\alpha(x, y) \geq 1$, then $(x, y) \in S(G)$.
Now, since F is ciric μ-graphic contractive mapping, so $d_q(Fx, Fy) \leq \mu(Q_q(x, y))$.
That is,

$$\alpha(x, y) \geq 1 \Rightarrow d_q(Fx, Fy) \leq \mu(Q_q(x, y)).$$

Let $\{x_n\} \subset B_{d_q}(x_0, r)$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then, $(x_n, x_{n+1}) \in S(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$. So by (ii) we have, $(x_n, x) \in S(G)$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n, x) \geq 1$.
Hence, all conditions of theorem 3.2 are satisfied and F has a fixed point.

Corollary 7: If (K, d_q) is a complete metric space endowed with a graph G and $F : K \to K$ a mapping. Suppose that the following assertions hold:
(i) F is Banach G-contraction on $B_{d_q}(x_0, r)$;
(ii) $(x_0, Fx_0) \in S(G)$ and $d_q(x_0, Fx_0) < (1-k)r$;
(iii) if $\{x_n\}$ is a sequence in $B_{d_q}(x_0, r)$ such that $(x_n, x_{n+1}) \in S(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in S(G)$ for all $n \in \mathbb{N}$.
Then F has a fixed point.

Corollary 8: If (K, d_q) is a complete metric space endowed with a graph G and $F : K \to K$ a mapping. Suppose that the following assertions hold:
(i) F is Banach G-contraction on K and there is $x_0 \in X$ such that $(x_0, Fx_0) \in S(G)$;
(ii) $(x_0, Fx_0) \in S(G)$ and $d_q(x_0, Fx_0) < (1-k)r$;
(iii) if $\{x_n\}$ is a sequence in K such that $(x_n, x_{n+1}) \in S(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in S(G)$ for all $n \in \mathbb{N}$.
Then F has a fixed point.

Conclusion

In conclusion, existence of fixed points for mappings satisfying graphic contraction conditions using Ciric-type mappings in an open ball of a complete metric space were proved. It was observed from the results that a point x^* exists in an open ball and F has a fixed point.

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References

[1] Abdeljawad T. (2013). Meir-Keeler $\alpha-$contractive fixed and common fixed point theorems. *Fixed Point Theory and Appl.* **19** (12), DOI: 10.1186/1687-1812-2013-19.

[2] ABDULLAH S., QASIM M., AQEEL S., MOHD SALMI M. & STOJAN R. (2021). Fixed point results for rational contraction in function weighted dislocated quasi-metric spaces with an application. *Advances in Difference Equations* **310**, 1-12. DOI: 10.1186/s13662-021-03458-X.

[3] ARSHAD M., SHOAIB A. & BEG I. (2013). Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space. *Fixed Point Theory and Appl.* **115**, 14 pages. DOI: 10.1186/1687-1812-2013-115.

[4] ARSHAD M., HUSSAIN A. & AZAM A. (2016). Fixed Point of $\alpha-$Geraghty contraction with applications. *U. P. B. Sci. Bull.* **78** (1), 57-66.

[5] GONCA D. G. & ISHAK A. (2024). Fixed point results for almost $(\zeta - \theta_p)-$contractions on quasi metric spaces and an application. *AIMS Mathematics* 9 (1), 763-774. doi: 10.3934/math.2024039.

[6] HUSSAIN N., AHMAD J. & AZAM A. (2014). Generalized fixed point theorems for multi-valued $\alpha - \psi-$contractive mappings. *J. Inequal. Appl.* **348**, 1-15. DOI: 10.1186/1472-6955-2014-348.

[7] HUSSAIN N., KARAPINAR E., SALIMI P. & AKBAR F. (2013). $\alpha-$admissible mappings and related fixed point theorems. *Journal of Inequalities and Applications.* **114**, 1-11. DOI: 10.1186/1029-242X-2013-114.

[8] JACHYMSKI J. (2008). The contraction principle for mappings on a metric space with a graph. *Proc. Amer. Math. Soc.* **136** (4), 1359-1373.

[9] KARAPINAR E. & SAMET B. (2012). Generalized $(\alpha - \psi)$ contractive type mappings and related fixed point theorems with applications. *Abstr. Appl. Anal.* Article id: 352981, 1-10. DOI: 10.1155/2012/352981.

[10] KUTBI M. A., ARSHAD M. & HUSSAIN N. (2014). On Modified $\alpha - \eta-$Contractive mappings. *Abstr. Appl. Anal.,* Article ID 947039, 1-8.DOI: 10.1155/2014/947039.

[11] MOHAMMADI B. & REZAPOUR S. (2013). On Modified $\alpha - \phi-$Contractions. *J. Adv. Math. Stud.,* **6** (2), 162-166.

[12] ROUHANI B. D. & MORADI S. (2014). On the existence and approximation of fixed points for Ciric type contractive mappings. *Quaestiones Mathematicae.* **37** (2), 179-189.

[13] SALIMI P., LATIF A. & HUSSAIN N. (2013). Modified $\alpha - \psi-$Contractive mappings with applications. *Fixed Point Theory Appl.,* 151, 1-19. DOI: 10.1186/1687-1812-2013-151.

[14] SAMET B., VETRO C. & VETRO P. (2012). Fixed point theorems for $\alpha - \psi-$contractive type mappings. *Nonlinear Anal.* **75** (4), 2154-2165. DOI:10.1016/j.na.2011.10.014.

[15] UR-RAHMAN M., ALI A., AZIZ O. & UR-RAHMON M. (2024). Fixed Point Theorems in Dislocated Quasi Metric and Dislocated Quasi $b-$Metric Spaces. *Communications in Nonlinear Analysis.* **1** (1), 1-10.