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**ON SPECIAL CASES OF OPIAL'S AND HARDY'S
INEQUALITIES**

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ABSTRACT

In this paper, we establish the relationship between Opial-type and Hardy-type integral inequalities which extend Antonio and Rauf Opial-type inequalities for convex function.

1. INTRODUCTION

The following interesting classical integral inequalities were stated and proved by Opial and Hardy respectively:

Theorem 1.1. *Let $x(t) \in C'[0, b]$ be such $x(0) = x(b) = 0$ and $x(t) > 0$ in $(0, b)$, then*

$$(1) \quad \int_a^b |x(t)x'(t)|dt \leq \frac{b}{4} \int_a^b (x'(t))^2 dt$$

where $\frac{b}{4}$ in the best possible constant. (See [9])

Theorem 1.2. *For $f(x) \geq 0$ and $p > 1$,*

$$(2) \quad \int_0^\infty \left[\frac{1}{x} f(t) dt \right]^p dx \leq q^p \int_0^\infty f^p(t) dt$$

where $q = \frac{p}{p-1}$ is the best possible constant. (see [5])

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In view of the usefulness of these inequalities in analysis and its applications generally, many authors have established the necessary and sufficient conditions on p, q, v, w for the Hardy-type inequality:

$$(3) \quad \left[\int_a^b |u(x)|^q w(x) dx \right]^{\frac{1}{q}} \leq C \left[\int_a^b |u'(x)|^p v(x) dx \right]^{\frac{1}{p}}$$

to hold, where C is a constant depending on p and q . See [7] and the reference therein.

Theorem 1.3. *Let g be continuous and non-decreasing on $[a, b]$, $0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and $f(x)$ be non-negative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then*

$$(4) \quad \left[\int_a^b g(x)^{\frac{\delta q}{p}} \left(\int_a^x f(t) dg(t) \right)^q dg(x) \right]^{\frac{1}{q}} \leq C(a, b, p, q, \delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x) \right]^{\frac{1}{p}}$$

where

$$(5) \quad C(a, b, p, q, \delta) = (-\delta)^{\frac{q(1-p)}{p}} \left(\frac{p}{q\delta + p} \right)^{\frac{p}{q}} g(b)^{\frac{q\delta+p}{p}} \left(g(b)^{-\delta} - g(a)^{-\delta} \right)^{\frac{q(p-1)}{p}} > 0$$

(See [2])

It was pointed out by Oguntuase (2009) that the constant $C(a, b, p, q, \delta)$ at the right hand side of (4) is wrong and stated the following:

Theorem 1.4. *Let g be a continuous and nondecreasing function on $[a, b]$, $0 \leq a < b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and let $f(x)$ be nonnegative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then,*

$$(6) \quad \left[\int_a^b g(x)^{\frac{q\delta}{p}} \left(\int_a^x f(t) dg(t) \right)^q dg(x) \right]^{\frac{1}{q}} \leq C(a, b, p, q, \delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f^p(x) dg(x) \right]^{\frac{1}{p}}$$

where

$$C(a, b, p, q, \delta) = (-\delta)^{\frac{1-p}{p}} \left(\frac{p}{q\delta + p} \right)^{\frac{1}{q}} \left(g(b)^{-\delta} - g(a)^{-\delta} \right)^{\frac{p-1}{p}} \left(g(b)^{\frac{q\delta+p}{p}} - g(a)^{\frac{q\delta+p}{p}} \right)^{\frac{1}{q}}$$

(See [8])

Some special cases of the result were obtained. The purpose of the work is to extend the work of Anthonio and Rauf (2015) with a view to obtain the relationship between Opial-type and Hardy-type classical inequalities.

2. MAIN RESULTS

Throughout this paper, we shall define: $h(x, t) = g(x)^\zeta f(t)^p g(t)^{p(1+\zeta)} d\mu(t)$ and $d\mu(t) = g(t)^{-(1+\zeta)} dg(t)$

The statement of the main results are as follows:

Lemma 2.1. *Let $h(x, t)$ be non negative, $x \geq 0, t \geq 0$ and $\mu \geq 0$ be non decreasing.*

Let $-\infty \leq 0 \leq x < \infty$, then the following holds:

$$(7) \quad \left[\int_a^x h(x, t) d\mu(t) \right]^{\frac{\nu}{p}} = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}}$$

Proof:

Let $f(t)$ and $g(x)$ are absolutely continuous functions.

$$(8) \quad \left[\int_a^x h(x, t) d\mu(t) \right]^{\frac{\nu}{p}} = \left[\int_a^x g(x)^\zeta f(t)^p g(t)^{(p-1)} g(t)^{(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}}$$

$$(9) \quad \left[\int_a^x h(x, t) d\mu(t) \right]^{\frac{\nu}{p}} = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}}$$

which complete the proof of the Lemma.

Lemma 2.2. *Let $h(x, t)$ be non negative, $t \geq 0, x \geq 0, p \geq 1$ and $\mu \geq 0$ be non decreasing.*

Let $-\infty \leq 0 \leq x < \infty$, then the following holds:

$$(10) \quad \left[\int_a^x d\mu(t) \right]^{\frac{(1-p)\nu}{p}} = \left[\frac{g(t)^{-\zeta}}{-\zeta} \Big|_a^x \right]^{\frac{(1-p)\nu}{p}} = [g(x) - g(a)]^{\frac{(p-1)\zeta\nu}{p}} (-\zeta)^{-\frac{(p-1)\nu}{p}}$$

Proof:

$$(11) \quad \begin{aligned} \left[\int_a^x d\mu(t) \right]^{\frac{(1-p)\nu}{p}} &= \left[\int_a^x g(t)^{-(1+\zeta)} dg(t) \right]^{\frac{(1-p)\nu}{p}} = \left[\frac{g(t)^{(-1-\zeta+1)}}{-1-\zeta+1} \Big|_a^x \right]^{\frac{(1-p)\nu}{p}} \\ &= \left[\frac{g(t)^{-\zeta}}{-\zeta} \Big|_a^x \right]^{\frac{(1-p)\nu}{p}} = [g(x)^\zeta - g(a)^\zeta]^{\frac{(p-1)\nu}{p}} (-\zeta)^{-\frac{(p-1)\nu}{p}} \end{aligned}$$

This completes the proof of the Lemma.

Lemma 2.3. *Suppose all the conditions of Lemma 2.2 hold, then we have:*

$$(12) \quad \left[\int_a^x h(x, t)^{\frac{1}{p}} d\mu(t) \right]^\nu = g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t) dg(t) \right]^\nu$$

Proof:

$$\begin{aligned} \left[\int_a^x h(x, t)^{\frac{1}{p}} d\mu(t) \right]^\nu &= \left[\int_a^x \left(g(x)^\zeta f(t)^p g(t)^{p(1+\zeta)} \right)^{\frac{1}{p}} g(x)^{-\zeta+1} dg(x) \right]^\nu \\ &= \left[\int_a^x \left(g(x)^{\frac{\zeta}{p}} f(t) g(t)^{(1+\zeta)} \right) g(x)^{-\zeta+1} dg(x) \right]^\nu \\ &= \left[\int_a^x g(x)^{\frac{\zeta}{p}} f(t) g(t)^{(1+\zeta)} g(x)^{-\zeta+1} dg(x) \right]^\nu \\ &= \left[\int_a^x g(x)^{\frac{\zeta}{p}} f(t) dg(x) \right]^\nu \\ (13) \quad &\Rightarrow \left[\int_a^x h(x, t)^{\frac{1}{p}} d\mu(t) \right]^\nu \geq g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t) dg(t) \right]^\nu \end{aligned}$$

The proof is completed.

Using the well known Jensen's inequality of the form: (see [3] and [4])

$$(14) \quad \int_a^x h(x, t) d\mu(s) \geq \left[\int_a^x d\mu(s) \right]^{1-p} \left[\int_a^x h(x, t)^{\frac{1}{p}} d\mu(s) \right]^p$$

Raising both sides of inequality (15) to power $\frac{\nu}{p}$ yields

$$(15) \quad \left[\int_a^x h(x, t) d\mu(s) \right]^{\frac{\nu}{p}} \geq \left[\int_a^x d\mu(s) \right]^{\frac{(1-p)\nu}{p}} \left[\int_a^x h(x, t)^{\frac{1}{p}} d\mu(s) \right]^\nu$$

Theorem 2.4. *Let $f(t)$ and $g(t)$ be a absolutely continuous function which is non-decreasing on $[a, b]$, $0 \leq a \leq b < \infty$. Suppose that $p \geq \nu \geq 1$, $\zeta > 0$ and $f(x)$ is Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Then,*

$$(16) \quad \left[\int_a^x f(t) dg(t) \right]^\nu \leq (-\zeta)^{\frac{(1-p)\nu}{p}} [g(x) - g(a)]^{\frac{(1-p)\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}}$$

Proof :

Multiply both side of (16) with $g(x)^{\frac{\zeta\nu}{p}}$ and by combining the results of Lemma 2.1, 2.2 and 2.3 in inequality (17), we get

$$(17) \quad g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}} \geq (-\zeta)^{\frac{(p-1)\nu}{p}} [g(x) - g(a)]^{\frac{(p-1)\zeta\nu}{p}} g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^b f(t) dg(t) \right]^\nu$$

that is

$$(18) \quad (-\zeta)^{\frac{(p-1)\nu}{p}} [g(x) - g(a)]^{\frac{(p-1)\zeta\nu}{p}} g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t) dg(t) \right]^\nu \leq g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}}$$

which implies

$$(19) \quad g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t) dg(t) \right]^\nu \leq (-\zeta)^{\frac{(1-p)\nu}{p}} [g(x) - g(a)]^{\frac{(1-p)\zeta\nu}{p}} g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}}$$

Integrating both sides of (20) with respect to $g(x)$ on $[a, b]$ and then raising both sides to power $\frac{p}{\nu}$ to obtain the following inequality:

$$(20) \quad \left[\int_a^b g(x)^{\frac{\zeta\nu}{p}} \left[\int_a^x f(t) dg(t) \right]^\nu dg(x) \right]^{\frac{p}{\nu}} \leq (-\zeta)^{\frac{(1-p)\nu}{p}} \left[\int_a^b [g(x) - g(a)]^{\frac{(1-p)\zeta\nu}{p}} g(x)^{\frac{\zeta\nu}{p}} \right. \\ \left. \times \left[\int_a^x f(t)^p g(x)^{(p-1)(1+\zeta)} dg(t) \right]^{\frac{\nu}{p}} dg(x) \right]^{\frac{p}{\nu}}$$

which is the type of the result in [2] and [8] generalization.

This has successfully suggested that Hardy-type and Opial-type of the two classical inequalities can be found in Jensen's inequality for convex function.

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