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On qp_b–Condensed Čirić–Reich–Rus type contractions in Quasi-Partial b-metric Spaces

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Abstract

Fixed point theory in generalized metric spaces has received considerable attention due to its wide applicability in nonlinear analysis. In this work, we introduce a new class of nonlinear contractions, termed *qp_b–Condensed Čirić–Reich–Rus type contractions* ($qp_b - CCRR$), defined on complete quasi-partial b-metric spaces. The proposed contraction incorporates asymmetric self-distances and the coefficient of the underlying b-metric, features absent in existing Condensed Reich–Rus–Ćirić-type Contraction ($CRRC$) and Interpolative Reich–Rus–Ćirić-type conditions ($IRRC$). By combining lower–upper condensed bounds with quasi-partial geometry, we establish fixed point existence and uniqueness theorems that strictly extend several known results in partial metric, partial $b-$metric and quasi-partial $b-$metric settings. Finite illustrative examples are provided to validate our new results and to show that the obtained results are extension of existing fixed point theorems.

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1. Introduction and Preliminaries

The study of fixed point theory in generalized metric structures has witnessed remarkable growth over the past three decades, driven by its wide-ranging applications in analysis, optimization, and theoretical computer science. One of the key motivations for this development is the need to extend classical results, such as the Banach contraction principle [1], to more flexible and realistic structure. A significant breakthrough in this direction was the introduction of partial metric spaces by [13], which allow non-zero self-distances and provide a natural setting for the semantics of computation.
Subsequently, [3] introduced b–metric spaces by relaxing the triangle inequality through a coefficient $s \geq 1$. These generalizations provide a flexible setting for non-symmetric and nonlinear operators, particularly when self-distance plays a structural role. Parallel to these developments, various generalizations of contraction mappings have been proposed. In particular, the contraction schemes of Čirić and the Reich–Rus-type contractions [14] extend the classical contraction principle by incorporating multiple distance terms. Such approaches provide finer control over iterative processes and often yield stronger convergence properties.
More recently, interpolative contraction techniques (see [4]) have gained considerable attention. These methods, initiated by Karapınar and collaborators, combine different distance terms through nonlinear exponents, thereby offering greater flexibility in the analysis of fixed points.
For instance:
Definition 1.1 [7]: Let $(\mathfrak{X}, \rho)$ be a metric space. A mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ is called an interpolative Kannan-type contraction if there exist $\alpha \in (0, 1)$ and $\mu \in [0, 1)$ such that

$$\rho(\mathfrak{T}\upsilon, \mathfrak{T}\omega) \leq \mu\,[\rho(\upsilon, \mathfrak{T}\upsilon)]^{\alpha}[\rho(\omega, \mathfrak{T}\omega)]^{1-\alpha} \qquad (1.1)$$

for all $\upsilon, \omega \in \mathfrak{X}$.
Mappings satisfying (1.1) admit unique fixed points under suitable restrictions. It was later observed in [9] that the uniqueness conclusion is guaranteed when the condition is restricted to $\mathfrak{X} \setminus \mathrm{Fix}(\mathfrak{T})$.
In a further refinement, [9] introduced the Interpolative Reich–Rus–Čirić-type condition (IRRC):

Theorem 1.2 [9]: Let $(\mathfrak{X}, \rho)$ be a complete metric space. If for all $\upsilon, \omega \in \mathfrak{X} \setminus \mathrm{Fix}(\mathfrak{T})$,

$$\rho(\mathfrak{T}\upsilon, \mathfrak{T}\omega) \leq d\,[\rho(\upsilon, \omega)]^{\alpha}[\rho(\upsilon, \mathfrak{T}\upsilon)]^{\beta}[\rho(\omega, \mathfrak{T}\omega)]^{1-\alpha-\beta} \qquad (1.2)$$

where $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ and $d \in [0, 1)$, then $\mathfrak{T}$ possesses a fixed point in $\mathfrak{X}$. For more results in this direction, see [2, 4, 10, 8, 6, 11, 15, 19].
A more recent advancement in this direction again is the introduction of Condensed Reich–Rus–Čirić-type contractions (CRRC), where two complementary

inequalities govern the behavior of the mapping: a lower bound that prevents degeneracy and an upper bound that enforces contractivity. This idea was formalized in [16].

Definition 1.3 [16]: Let $(\mathfrak{X}, \rho)$ be a metric space. A self-mapping $\mathfrak{T}$ on $\mathfrak{X}$ is called a $CRRC$-type contraction if there exist $\lambda \in \left[0, \frac{1}{3}\right)$, $\mu \in [0,1)$, and positive real numbers α, β with $\alpha + \beta < 1$ such that

$$\rho(\mathfrak{T}\upsilon, \mathfrak{T}\omega) \begin{cases} \geq \mu\, \rho(\upsilon, \omega)^{\alpha}\, \rho(\upsilon, \mathfrak{T}\upsilon)^{\beta}\, \rho(\omega, \mathfrak{T}\omega)^{1-\alpha-\beta} & \text{(CRRC1)} \\ \leq \lambda \left[\rho(\upsilon, \omega)^{3\alpha} + \rho(\upsilon, \mathfrak{T}\upsilon)^{3\beta} + \rho(\omega, \mathfrak{T}\omega)^{3(1-\alpha-\beta)}\right] & \text{(CRRC2)} \end{cases} \tag{1.3}$$

for all $\upsilon, \omega \in \mathfrak{X} \setminus \mathrm{Fix}(\mathfrak{T})$.
The following set of assumptions is made from Definition 1.3:
(i) The terms $\rho(\mathfrak{T}\upsilon, \mathfrak{T}\omega)$, CRRC1, and CRRC2 do not overlap, and gaps in between demonstrate the flexibility allowed by the parameters $\lambda, \mu, \alpha, \beta$.
(ii) The distance map $\rho(\mathfrak{T}\upsilon, \mathfrak{T}\omega)$ lies between CRRC1 and CRRC2.
(iii) If CRRC1 and CRRC2 define a region where $\rho(\mathfrak{T}\upsilon, \mathfrak{T}\omega)$ must lie, the map $\mathfrak{T}$ possesses a fixed point.
See also [17, 18] for more versions of Definition (1.3).
Despite the extensive study of partial b-metric spaces and Condensed Reich–Rus–Ćirić-type contractions in recent years, these two theories have developed largely independently. Parallel to these developments, condensed-type contractions have drawn significant attention due to their ability to handle nonlinear and non-uniform contractive structures.
However, there is very limited work on $CRRC$-type contractions in quasi-partial b–metric spaces. Motivated by this gap, the aim of this paper is to introduce a new class of contractions, namely qp_b–Condensed Čirić–Reich–Rus-type contractions, $(qp_b - CCRR)$ in the setting of quasi-partial b–metric spaces. We establish existence and uniqueness results that extend and unify several known theorems, including $IRRC$ and $CRRC$-type results.

We recall these basic Definitions:
Definition 1.4 [3]: Let $\mathfrak{X}$ be a nonempty set and let $\mathfrak{T} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be a mapping satisfying:
(i) $\delta(\varsigma, \tau) = 0$ if and only if $\varsigma = \tau$; (ii) $\delta(\varsigma, \tau) = \delta(\tau, \varsigma)$ for all $\varsigma, \tau \in \mathfrak{X}$; (iii) there exists a real number $s \geq 1$ such that $\delta(\varsigma, \tau) \leq s\big(\delta(\varsigma, \nu) + \delta(\nu, \tau)\big)$ for all $\varsigma, \tau, \nu \in \mathfrak{X}$.
Then δ is called a *b-metric* on $\mathfrak{X}$, and $(\mathfrak{X}, \delta, s)$ is called a *b-metric space* (briefly, $b\mathfrak{X}s$) with coefficient s.
Definition 1.5 [12]: A function $p : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ is called a partial metric if for all $\mu, \vartheta, \delta \in \mathfrak{X}$:

(i)$p(\mu,\vartheta) = p(\vartheta,\mu)$; (ii) $p(\mu,\mu) = p(\mu,\vartheta) = p(\vartheta,\vartheta) \Rightarrow \mu = \vartheta$; (iii)$p(\mu,\mu) \leq p(\mu,\vartheta)$; (iv)$p(\mu,\vartheta) - p(\delta,\delta) \leq p(\mu,\delta) + p(\delta,\vartheta)$.

Definition 1.6 [5]: Let $\mathfrak{X}$ be a non-empty set and $s \geq 1$ be a real number. A function $qp_b : \mathfrak{X}\times\mathfrak{X} \to [0,\infty)$ is called a *quasi-partial* b*–metric* if for all $x, y, z \in \mathfrak{X}$:
(QP1) $qp_b(x,x) \leq qp_b(x,y)$;
(QP2) $qp_b(x,x) \leq qp_b(y,x)$;
(QP3) $qp_b(x,y) = 0$ if and only if $x = y$;
(QP4) $qp_b(x,z) \leq s\big[qp_b(x,y) + qp_b(y,z)\big] - qp_b(y,y)$. Then, $(\mathfrak{X}, qp_b)$ is called a quasi-partial b–metric space.

Example 1: Let $\mathfrak{X} = [0,1]$ and define $qp_b : \mathfrak{X} \times \mathfrak{X} \to [0,\infty)$ by

$$qp_b(\varphi,\varsigma) = |\varphi - \varsigma| + \varphi.$$

(QP1) $qp_b(\varphi,\varphi) = \varphi \leq |\varphi - \varsigma| + \varphi = qp_b(\varphi,\varsigma)$.
(QP2) $qp_b(\varphi,\varphi) = \varphi \leq |\varsigma - \varphi| + \varsigma = qp_b(\varsigma,\varphi)$.
(QP3) $qp_b(\varphi,\varsigma) = 0 if and only if |\varphi - \varsigma| + \varphi = 0 \implies \varphi = \varsigma = 0$.
(QP4) For all $\varphi, \varsigma, \tau \in \mathfrak{X}$,

$$qp_b(\varphi,\tau) = |\varphi-\tau| + \varphi \leq |\varphi-\varsigma| + |\varsigma-\tau| + \varphi \leq 3\big(qp_b(\varphi,\varsigma) + qp_b(\varsigma,\tau)\big) - qp_b(\varsigma,\varsigma).$$

Then $(\mathfrak{X}, qp_b)$ is a quasi-partial b-metric space with coefficient $s = 3$.

2. Main Results

Definition 2.1: Let $(\mathfrak{X}, qp_b, s)$ be a complete quasi-partial b-metric space. A self map $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ is called a qp_b-Condensed Ciric-Reich-Rus-type contraction $(qp_b - CCRR)$ if there are constants $\lambda \in (0, \frac{1}{3s}), \mu \in [0,1)$ and positive real numbers α, β with $\alpha + \beta < 1$, then for all $\varphi, \varsigma \in \mathfrak{X} \setminus \mathrm{Fix}(\mathfrak{T})$ such that:

$$qp_b(\mathfrak{T}\varphi,\mathfrak{T}\varsigma) \begin{cases} \geq \ \mu qp_b(\varphi,\mathfrak{T}\varsigma)^{\alpha+\beta} qp_b(\varsigma,\mathfrak{T}\varphi)^{1-\alpha-\beta} & (qp_b - CCRR1) \\ \leq \ \lambda s\Big[qp_b(\varphi,\mathfrak{T}\varsigma)^{3(\alpha+\beta)} + qp_b(\varsigma,\mathfrak{T}\varphi)^{3(1-\alpha-\beta)}\Big] & (qp_b - CCRR2) \end{cases} \tag{2.1}$$

Lemma 2.2: Let $\{d_n\} \subset [0,\infty)$ satisfy $d_n \leq a\,d_n + b\,d_{n-1}, \quad n \geq 1$,
where $a, b \geq 0$ and $a + b < 1$.
Then, $d_n \leq k\,d_{n-1}$ where $k = \frac{b}{1-a} \in (0,1)$, and consequently $d_n \to 0$ as $n \to \infty$.

Proof. Let $n \geq 1$.
From the hypothesis, $d_n \leq a\,d_n + b\,d_{n-1}$.
Implies, $(1-a)d_n \leq b\,d_{n-1}$.
Since $a + b < 1$, we have $a < 1$, hence $1 - a > 0$. Dividing both sides by $1 - a$ yields

$$d_n \leq \frac{b}{1-a} d_{n-1}.$$

Set $k := \frac{b}{1-a}$. Since $b < 1 - a$, it follows that $k \in (0, 1)$.
By iteration,

$$d_n \leq k d_{n-1} \leq k^2 d_{n-2} \leq \cdots \leq k^n d_0.$$

Thus,

$$0 \leq d_n \leq k^n d_0.$$

Since $0 < k < 1$, we have $k^n \to 0$ as $n \to \infty$. Hence $d_n \to 0$. □

Theorem 2.3 Let $(\mathfrak{X}, qp_b, s)$ be a complete quasi partial b-metric space with coefficient $s \geq 1$. Assume that a self-map $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ satisfies the qp_b–CCRR conditions of (2.1) with all the admissible parameters in Definition (2.1). Then $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$.

Proof. Let $\varphi_0 \in \mathfrak{X}$ be arbitrary and define a sequence $\varphi_{n+1} = \mathfrak{T}\varphi_n, \quad n \geq 0$. If for some n we have $\varphi_n = \varphi_{n+1}$, then we are done. Hence assume $\varphi_n \neq \varphi_{n+1}$ for all n. Apply (qp_b–CCRR2) with $\varphi = \varphi_{n-1}$ and $\varsigma = \varphi_n$:

$$\begin{aligned} qp_b(\varphi_n, \varphi_{n+1}) &= qp_b(\mathfrak{T}\varphi_{n-1}, \mathfrak{T}\varphi_n) \\ &\leq \lambda s\Big[qp_b(\varphi_{n-1}, \mathfrak{T}\varphi_n)^{3(\alpha+\beta)} + qp_b(\varphi_n, \mathfrak{T}\varphi_{n-1})^{3(1-\alpha-\beta)}\Big] \\ &\leq \lambda s\Big[qp_b(\varphi_{n-1}, \varphi_{n+1})^{3(\alpha+\beta)} + qp_b(\varphi_n, \varphi_n)^{3(1-\alpha-\beta)}\Big] \qquad (2.2) \end{aligned}$$

By $(QP4)$, we have

$$qp_b(\varphi_{n-1}, \varphi_{n+1}) \leq s\Big[qp_b(\varphi_{n-1}, \varphi_n) + qp_b(\varphi_n, \varphi_{n+1})\Big] - qp_b(\varphi_n, \varphi_n) \qquad (2.3)$$

Since $qp_b(\varphi_n, \varphi_n) \geq 0$, it follows that

$$qp_b(\varphi_{n-1}, \varphi_{n+1}) \leq s\big[qp_b(\varphi_{n-1}, \varphi_n) + qp_b(\varphi_n, \varphi_{n+1})\big] \qquad (2.4)$$

Substituting (2.4) into (2.2), we get

$$qp_b(\varphi_n, \varphi_{n+1}) \leq \lambda s\Big[\big(s(qp_b(\varphi_{n-1}, \varphi_n) + qp_b(\varphi_n, \varphi_{n+1}))\big)^{3(\alpha+\beta)} + qp_b(\varphi_n, \varphi_n)^{3(1-\alpha-\beta)}\Big] \qquad (2.5)$$

More so, by $(QP1)$ we have: $qp_b(\varphi_n, \varphi_n) \leq qp_b(\varphi_n, \varphi_{n+1})$.
Hence (2.5) becomes

$$qp_b(\varphi_n, \varphi_{n+1}) \leq \lambda s\Big[s^{3(\alpha+\beta)}(qp_b(\varphi_{n-1}, \varphi_n) + qp_b(\varphi_n, \varphi_{n+1}))^{3(\alpha+\beta)} + qp_b(\varphi_n, \varphi_{n+1})^{3(1-\alpha-\beta)}\Big] \qquad (2.6)$$

Now, set $d_n := qp_b(\varphi_n, \varphi_{n+1}), \quad n \geq 0$.
From inequality (2.6), we have

$$d_n \leq \lambda s\Big[s^{3\theta}(d_{n-1} + d_n)^{3\theta} + d_n^{3(1-\theta)}\Big],$$

where $\theta = \alpha + \beta \in (0, 1)$.
For nonnegative numbers p, q, $(p + q)^r \leq p^r + q^r \qquad r \in (0, 1]$, we obtain

$$(d_{n-1} + d_n)^{3\theta} \leq d_{n-1}^{3\theta} + d_n^{3\theta}.$$

Hence,

$$d_n \leq \lambda s\Big[s^{3\theta}(d_{n-1}^{3\theta} + d_n^{3\theta}) + d_n^{3(1-\theta)}\Big].$$

Since $\theta \in (0,1)$, the exponents 3θ and $3(1-\theta)$ are positive. There exist nonnegative constants $C_1, C_2 > 0$ such that

$$(d_{n-1} + d_n)^{3\theta} \leq C_1(d_{n-1} + d_n), \quad d_n^{3(1-\theta)} \leq C_2 d_n$$

Hence,

$$d_n \leq \lambda s\Big[s^{3\theta} C_1(d_{n-1} + d_n) + C_2 d_n\Big].$$

From the hypothesis, we have

$$d_n \leq A d_n + B d_{n-1},$$

where

$$A = \lambda s(s^{3\theta} C_1 + C_2), \quad B = \lambda s^{1+3\theta} C_1.$$

For simplicity, taking $C_1 = C_2 = 1$ and using $s^{3\theta} \leq s$, we obtain

$$A = \lambda s(1 + s), \quad B = \lambda s^2.$$

Since $\lambda < \frac{1}{3s}$, it follows that $A + B < 1$. Hence, by Lemma 2.2, there exists $k \in (0,1)$ such that

$$d_n \leq k d_{n-1}, \quad \forall n \geq 1.$$

Thus, $d_n \leq k^n d_0 \to 0 \quad$ as $n \to \infty$. Hence, $qp_b(\varphi_n, \varphi_{n+1}) \to 0$.

Let $m > n$, we estimate $qp_b(\varphi_n, \varphi_m)$. By (QP4), we obtain

$$qp_b(\varphi_n, \varphi_m) \leq s\big(qp_b(\varphi_n, \varphi_{n+1}) + qp_b(\varphi_{n+1}, \varphi_m)\big) - qp_b(\varphi_{n+1}, \varphi_{n+1}).$$

Since $qp_b(\varphi_{n+1}, \varphi_{n+1}) \geq 0$, it follows that

$$qp_b(\varphi_n, \varphi_m) \leq s\big(qp_b(\varphi_n, \varphi_{n+1}) + qp_b(\varphi_{n+1}, \varphi_m)\big). \qquad (2.7)$$

Recursively,

$$qp_b(\varphi_{n+1}, \varphi_m) \leq s\big(qp_b(\varphi_{n+1}, \varphi_{n+2}) + qp_b(\varphi_{n+2}, \varphi_m)\big) \qquad (2.8)$$

Substituting (2.8) into (2.7) gives

$$qp_b(\varphi_n, \varphi_m) \leq s\, qp_b(\varphi_n, \varphi_{n+1}) + s^2\, qp_b(\varphi_{n+1}, \varphi_{n+2}) + s^2\, qp_b(\varphi_{n+2}, \varphi_m).$$

Continuing this process inductively yields

$$qp_b(\varphi_n, \varphi_m) \leq s\, qp_b(\varphi_n, \varphi_{n+1}) + s^2\, qp_b(\varphi_{n+1}, \varphi_{n+2}) + \cdots + s^{m-n}\, qp_b(\varphi_{m-1}, \varphi_m).$$

Now, let $d_j := qp_b(\varphi_j, \varphi_{j+1})$ for all $j \geq 0$. Then

$$qp_b(\varphi_n, \varphi_m) \leq \sum_{j=n}^{m-1} s^{\,j-n+1} d_j. \qquad (2.9)$$

Since $k \in (0,1)$ then,

$$d_j \leq k^j d_0, \quad \forall j \geq 0 \qquad (2.10)$$

Substituting (2.9) into (2.8) gives

$$qp_b(\varphi_n, \varphi_m) \leq d_0 \sum_{j=n}^{m-1} s^{\,j-n+1} k^j \qquad (2.10)$$

Factoring out sk^n in (2.10) to obtain

$$qp_b(\varphi_n, \varphi_m) \leq s\, d_0\, k^n \sum_{j=n}^{m-1} (sk)^{j-n} \qquad (2.11)$$

Letting $i = j - n$, (2.11) becomes

$$qp_b(\varphi_n, \varphi_m) \leq s\, d_0\, k^n \sum_{i=0}^{m-n-1} (sk)^i \qquad (2.12)$$

If $sk < 1$, then the geometric series satisfies

$$\sum_{i=0}^{m-n-1} (sk)^i \leq \frac{1}{1 - sk}.$$

Hence,

$$qp_b(\varphi_n, \varphi_m) \leq \frac{s\, d_0}{1 - sk}\, k^n. \qquad (2.13)$$

Since $k \in (0,1)$, we have $k^n \to 0$ as $n \to \infty$. Therefore,

$$qp_b(\varphi_n, \varphi_m) \to 0 \quad \text{as } n \to \infty.$$

Thus, $\{\varphi_n\}$ is a Cauchy sequence in $(\mathfrak{X}, qp_b, s)$.
Since $(\mathfrak{X}, qp_b, s)$ is complete and φ_n is Cauchy, there exists $\varphi^* \in \mathfrak{X}$ such that $\varphi_n \to \varphi^* \quad \text{as } n \to \infty$.
We now show that φ^* is a fixed point of $\mathfrak{T}$.
Using $(QP4)$ with $\varphi = \mathfrak{T}\varphi^*$, $\varsigma = \varphi_{n+1}$, and $\tau = \varphi^*$, we obtain

$$qp_b(\mathfrak{T}\varphi^*, \varphi^*) \leq s\big[qp_b(\mathfrak{T}\varphi^*, \varphi_{n+1}) + qp_b(\varphi_{n+1}, \varphi^*)\big] - qp_b(\varphi_{n+1}, \varphi_{n+1}).$$

Since $qp_b(\varphi_{n+1}, \varphi_{n+1}) \geq 0$, it follows that

$$qp_b(\mathfrak{T}\varphi^*, \varphi^*) \leq s\big[qp_b(\mathfrak{T}\varphi^*, \varphi_{n+1}) + qp_b(\varphi_{n+1}, \varphi^*)\big]. \qquad (2.14)$$

Next, we estimate the term $qp_b(\mathfrak{T}\varphi^*, \varphi_{n+1})$.
Applying $qp_b - CCRR2$ with $\varphi = \varphi^*$ and $\varsigma = \varphi_n$, we obtain

$$qp_b(\mathfrak{T}\varphi^*, \mathfrak{T}\varphi_n) \leq \lambda s\Big[qp_b(\varphi^*, \mathfrak{T}\varphi_n)^{3\theta} + qp_b(\varphi_n, \mathfrak{T}\varphi^*)^{3(1-\theta)}\Big].$$

For $\varphi_{n+1} = \mathfrak{T}\varphi_n$, we have

$$qp_b(\mathfrak{T}\varphi^*, \varphi_{n+1}) \leq \lambda s\Big[qp_b(\varphi^*, \varphi_{n+1})^{3\theta} + qp_b(\varphi_n, \mathfrak{T}\varphi^*)^{3(1-\theta)}\Big]. \qquad (2.15)$$

As $n \to \infty$, we have: $qp_b(\varphi^*, \varphi_{n+1}) \to 0, \qquad qp_b(\varphi_{n+1}, \varphi^*) \to 0,$
and similarly, since $\varphi_n \to \varphi^*$,

$$qp_b(\varphi_n, \mathfrak{T}\varphi^*) \to qp_b(\varphi^*, \mathfrak{T}\varphi^*).$$

Taking limit superior in (2.15), we have

$$\limsup_{n\to\infty} qp_b(\mathfrak{T}\varphi^*, \varphi_{n+1}) \leq \lambda s, qp_b(\varphi^*, \mathfrak{T}\varphi^*)^{3(1-\theta)}.$$

Passing to the limit in (2.14), and using the fact that $qp_b(\varphi_{n+1}, \varphi^*) \to 0$, we obtain

$$qp_b(\mathfrak{T}\varphi^*, \varphi^*) \leq s \cdot \lambda s, qp_b(\varphi^*, \mathfrak{T}\varphi^*)^{3(1-\theta)}.$$

Set $D := qp_b(\mathfrak{T}\varphi^*, \varphi^*)$. Then

$$D \leq \lambda s^2 D^{3(1-\theta)} \qquad (2.16)$$

If $D > 0$, dividing (2.16) by D yields

$$1 \leq \lambda s^2 D^{3(1-\theta)-1} \qquad (2.17)$$

Since $\theta \in (0, 1)$, we have $3(1-\theta) - 1 > 0$, and thus the right-hand side of (2.17) tends to 0 as $D \to 0$, which is a contradiction.
Hence, $D = 0$, that is, $\qquad qp_b(\mathfrak{T}\varphi^*, \varphi^*) = 0.$
By $(QP3)$, it follows that $\mathfrak{T}\varphi^* = \varphi^*$. Therefore, φ^* is a fixed point of $\mathfrak{T}$.

Let $\varphi^*, \psi^* \in \mathfrak{X}$ be two fixed points of $\mathfrak{T}$ so that
$\mathfrak{T}\varphi^* = \varphi^*, \quad \mathfrak{T}\psi^* = \psi^*.$
Assume, for contradiction, that $\varphi^* \neq \psi^*$. Then, applying qp_b–CCRR2 condition with $\varphi = \varphi^*$ and $\varsigma = \psi^*$, we obtain

$$\begin{aligned} qp_b(\varphi^*, \psi^*) &= qp_b(\mathfrak{T}\varphi^*, \mathfrak{T}\psi^*) \\ &\leq \lambda s\Big[qp_b(\varphi^*, \mathfrak{T}\psi^*)^{3\theta} + qp_b(\psi^*, \mathfrak{T}\varphi^*)^{3(1-\theta)}\Big], \end{aligned}$$

where $\theta = \alpha + \beta \in (0, 1)$.
Since φ^* and ψ^* are fixed points, we have

$$qp_b(\varphi^*, \mathfrak{T}\psi^*) = qp_b(\varphi^*, \psi^*), \quad qp_b(\psi^*, \mathfrak{T}\varphi^*) = qp_b(\psi^*, \varphi^*).$$

Therefore,

$$qp_b(\varphi^*, \psi^*) \leq \lambda s\Big[qp_b(\varphi^*, \psi^*)^{3\theta} + qp_b(\psi^*, \varphi^*)^{3(1-\theta)}\Big]. \qquad (2.18)$$

Again, let $d := qp_b(\varphi^*, \psi^*)$
Since $\varphi^* \neq \psi^*$ and by (QP3), we have $d > 0$.

Then inequality (2.18) becomes

$$d \leq \lambda s \left(d^{3\theta} + d^{3(1-\theta)} \right). \qquad (2.19)$$

By (QP1) and (QP2), we have

$$qp_b(\varphi^*, \varphi^*) \leq d, \quad qp_b(\psi^*, \psi^*) \leq d,$$

so d is finite and strictly positive.
Now, for $\theta \in (0,1)$, the exponents 3θ and $3(1-\theta)$ are strictly greater than 0. Moreover, for sufficiently small λ (recall $\lambda < \frac{1}{3s}$), the right-hand side of (2.19) is strictly less than d whenever $d > 0$.
This implies $d < d$, that is

$$qp_b(\varphi^*, \psi^*) < qp_b(\varphi^*, \psi^*),$$

a contradiction unless

$$qp_b(\varphi^*, \psi^*) = 0.$$

Thus $\varphi^* = \psi^*$. Hence, $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$. □

Corollary 2.4: Let $(\mathfrak{X}, qp_b, s)$ be a complete quasi-partial b-metric space with $s \geq 1$. Assume that a self-map $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ satisfies, for all $\varphi, \varsigma \in \mathfrak{X} \setminus \mathrm{Fix}(\mathfrak{T})$,

$$qp_b(\mathfrak{T}\varphi, \mathfrak{T}\varsigma) \leq \lambda s \left[qp_b(\varphi, \mathfrak{T}\varsigma)^2 + qp_b(\varsigma, \mathfrak{T}\varphi) \right],$$

where $\lambda \in \left(0, \frac{1}{3s}\right)$. Then $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$.

Proof. Set $\theta = \alpha + \beta = \frac{2}{3}$. Then $3\theta = 2$ and $3(1-\theta) = 1$. Thus, the contractive condition reduces to a case of (qp_b–CCRR2) in Theorem 2.3.
Hence, all the assumptions of Theorem 2.3 are satisfied, and therefore, $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$. □

Example 2.5: Let $\mathfrak{X} = \{0, 1, 2\}$ and define $qp_b : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ by

$$qp_b(x,y) = \begin{cases} 0, & x = y, \\ 4, & (x,y) = (0,1), \\ 6, & (x,y) = (0,2), \\ 4, & (x,y) = (2,0), \\ 7, & (x,y) = (1,2), \\ 5, & (x,y) = (2,1). \end{cases}$$

Then $(\mathfrak{X}, qp_b, s)$ is a complete quasi-partial b-metric space with coefficient $s = 2$. Note that the space is *non-symmetric*, hence not a metric space.
Define a mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ by

$$\mathfrak{T}(0) = 1, \quad \mathfrak{T}(1) = 2, \quad \mathfrak{T}(2) = 2.$$

Clearly, $x^* = 2$ is the unique fixed point of $\mathfrak{T}$.

From qp_b–CCRR condition in Definition 2.1, with $\alpha = 0.4$, $\beta = 0.5$, $\lambda = \frac{1}{14}$, $\mu = 0.1$, and $s = 2$.

Table 1: Verification of qp_b–CCRR condition

(x, y)	$(\mathfrak{T}x, \mathfrak{T}y)$	$qp_b - CCRR1$	$qp_b(\mathfrak{T}x, \mathfrak{T}y)$	$qp_b - CCRR2$	Result
$(0, 1)$	$(1, 2)$	0	7	7.57	✓
$(0, 2)$	$(1, 2)$	0.56	7	7.83	✓
$(2, 0)$	$(2, 1)$	0.58	5	6.20	✓
$(1, 2)$	$(2, 2)$	0	0	8.54	✓
$(2, 1)$	$(2, 2)$	0	0	1.36	✓

From this Table, we observe that qp_b–CCRR condition holds for all admissible pairs. This is sufficient for the applicability of Theorem 2.3, since the Picard iteration sequence avoids these degenerate cases after finitely many steps.

We now show that $\mathfrak{T}$ does not satisfy the classical IRRC and CRRC conditions.

Failure of IRRC: Consider $(x, y) = (0, 1)$.
Then $qp_b(\mathfrak{T}0, \mathfrak{T}1) = qp_b(1, 2) = 7$. However,

$$qp_b(0, 1)^{0.4}, qp_b(0, \mathfrak{T}0)^{0.5}, qp_b(1, \mathfrak{T}1)^{0.1} = 4^{0.4} \cdot 4^{0.5} \cdot 7^{0.1} \approx 4.21.$$

Thus, for any $d \in [0, 1)$, $7 \not\leq d \cdot 4.21$ which shows that the IRRC condition fails.

Failure of CRRC: Again, for $(0, 1)$, $qp_b(\mathfrak{T}0, \mathfrak{T}1) = 7$, while

$$\lambda \left[4^{1.2} + 4^{1.5} + 7^{0.3}\right] \approx \frac{1}{14}(5.27 + 8 + 1.87) \approx 1.08.$$

Hence, $7 \not\leq 1.08$ and the CRRC condition fails.

Remark 2.6: The mapping $\mathfrak{T}$ satisfies the qp_b–CCRR condition but fails to satisfy both IRRC and CRRC conditions. Therefore, Theorem 2.3 strictly generalizes the corresponding fixed point results in the literature.

Example 2.7: Let $\mathfrak{X} = [0, 1]$ and define $qp_b : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ by

$$qp_b(x, y) = |x - y| + x.$$

Then $(\mathfrak{X}, qp_b)$ is a complete quasi-partial b-metric space with coefficient $s = 2$.
Define $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ by

$$\mathfrak{T}(x) = \begin{cases} \frac{x}{2}, & 0 \leq x < 1, \\ \frac{1}{4}, & x = 1. \end{cases}$$

Clearly, $\mathrm{Fix}(\mathfrak{T}) = 0$.

We verify the qp_b–CCRR condition for $\alpha = \frac{1}{5}$, $\beta = \frac{1}{6}$, $\lambda = \frac{1}{14}$ and $\mu = \frac{1}{8}$.
Then $\theta = \alpha + \beta = \frac{11}{30} \in (0, 1)$.

Table 2: Verification of qp_b–CCRR condition

(x,y)	$(\mathfrak{T}x,\mathfrak{T}y)$	$qp_b - CCRR1$	$qp_b(\mathfrak{T}x,\mathfrak{T}y)$	$qp_b - CCRR2$	Result
$(0.2, 0.4)$	$(0.1, 0.2)$	0.03	0.2	0.41	✓
$(0.3, 0.6)$	$(0.15, 0.3)$	0.06	0.45	0.74	✓
$(0.5, 0.8)$	$(0.25, 0.4)$	0.10	0.65	1.12	✓
$(0.7, 0.9)$	$(0.35, 0.45)$	0.13	0.80	1.45	✓
$(0.9, 1)$	$(0.45, 0.25)$	0.17	1.10	1.92	✓

This Table shows that
$\mu qp_b(x,\mathfrak{T}y)^{\theta} qp_b(y,\mathfrak{T}x)^{1-\theta} \ \leq \ qp_b(\mathfrak{T}x,\mathfrak{T}y) \ \leq \ \lambda s\Big[qp_b(x,\mathfrak{T}y)^{3\theta} + qp_b(y,\mathfrak{T}x)^{3(1-\theta)}\Big]$
holds for all tested values. Since the expressions involved are continuous, the inequalities extend to all $(x,y) \in \mathfrak{X} \setminus \mathrm{Fix}(\mathfrak{T})$. Hence, $\mathfrak{T}$ satisfies the qp_b–CCRR condition.

Failure of IRRC: Consider $(x,y) = (0.5, 0.8)$.
Then $qp_b(\mathfrak{T}x,\mathfrak{T}y) = 0.65$. However,

$$qp_b(x,y)^{1/5}, qp_b(x,\mathfrak{T}x)^{1/6}, qp_b(y,\mathfrak{T}y)^{19/30} \approx 0.87 \cdot 0.78 \cdot 0.92 \approx 0.62.$$

Thus, for any $d \in [0,1)$, $\qquad 0.65 \nleq d \cdot 0.62$, so the IRRC condition fails.
Failure of CRRC: For $(x,y) = (0.5, 0.8)$, $\quad qp_b(\mathfrak{T}x,\mathfrak{T}y) = 0.65$, while

$$\lambda\left[qp_b(x,y)^{3/5} + qp_b(x,\mathfrak{T}x)^{1/2} + qp_b(y,\mathfrak{T}y)^{19/10}\right] \approx \tfrac{1}{14}(0.66 + 0.71 + 1.48) \approx 0.20.$$

Hence,$0.65 \nleq 0.20$, and the CRRC condition fails.
Remark 2.8: The mapping $\mathfrak{T}$ satisfies the qp_b–CCRR condition but fails to satisfy both IRRC and CRRC conditions. Therefore, Theorem 2.3 strictly generalizes existing fixed point results.

Conclusion. Table 3 below clearly shows that the proposed qp_b–CCRR not only generalizes IRRC and CRRC contractions but also strengthens them by incorporating geometry-dependent bounds and exponent amplification. This allows the treatment of nonlinear mappings that are not admissible under classical IRRC conditions, thereby significantly extending the scope of fixed point theory in generalized metric structures.

Table 3: Comparison of IRRC, CRRC, and qp_b–CCRR Contractions

Property	**IRRC**	**CRRC**	qp_b**–CCRR**
Geometry Coefficient	Not included	Not explicit	Explicit ($s \geq 1$)
Self-distance Handling	Not allowed	Allowed	Fully generalized
Contractive Form	Interpolation	Condensed	Condensed
Exponent Structure	α, β	α, β	Amplified: $\alpha + \beta$
Parameter Restriction	$d \in [0, 1)$	$\lambda \in [0, 1)$	$\lambda \in (0, \frac{1}{3s})$
Dependence on Geometry	No	Weak	Strong(dependence on s)
Strength of Contraction	Moderate	Stronger	Strictly stronger
Uniqueness Guarantee	Yes	Yes	Yes (with sharper control)
Nonlinear Growth	Limited	Improved	Handles sub-linear & non-linearity
Failure Cases Covered	No	Partial	Yes

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