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On Polynomials Derived from Topological Indices of Special Graphs

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Abstract

Polynomials associated with topological indices are defined from graph-based indices with their coefficients and exponents derived from these values. This paper presents formulae for such polynomials for cycle C_n, wheel W_n, path P_n, complete graph K_n and complete bipartite graph $K_{m,n}$. It is further shown that a graph is a closed path if and only if its first and second Zagreb indices coincide.

1. Introduction and Preliminaries

The pair $G(V, E)$ consisting of sets of vertices (V) and edges (E) is called a graph with the cardinality of V as order of the graph G and cardinality of E as the size of G [2]. A topological index is a number obtained from a graph, it is also known as graph invariant as it remains fixed under graph homomorphism [3]. Topological indices are very vital in mathematical chemistry as they are used to predict the properties of chemical compounds through their molecular graphs, such properties are stability, boiling point, viscosity, strain and entropy [11].

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Wiener's index [13] was the first index to be studied, it was introduced by Harold Wiener in 1947 while working on the structural determination of Paraffin boiling points. The Wiener's index of a graph G is denoted as $W(G)$ and defined as

$$W(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d(u,v). \tag{1.1}$$

With $d(u,v)$ the distance between vertices u and v in a connected graph, where distance in this case refers to the number of few edges between the vertices [4], i.e we take the shortest path between them.
Gutman and Trinajstic [6] introduced the first and second Zagreb indices in 1972. They defined the indices in terms of the degree of a vertex. The degree of a vertex v denoted as d_v is the number of edges incident with it [4], i.e the number of edges linking v with other vertices in the graph. Below are the two Zagreb indices; the first is defined as

$$M_1(G) = \sum_{v \in V(G)} (d_v)^2 \tag{1.2}$$

and the second is define as

$$M_2(G) = \sum_{uv \in E(G)} d_u \, . \, d_v, \tag{1.3}$$

as seen in [9] the first Zagreb index is also written as

$$M_1(G) = \sum_{uv \in E(G)} [d_u + d_v]. \tag{1.4}$$

Shirdel *et al.* [12] in 2013 also looked at the formula for computing the hyper-Zagreb index of graphs which they gave as

$$HM(G) = \sum_{uv \in E(G)} [d_u + d_v]^2 \tag{1.5}$$

and the redefined third Zagreb index is also shown in [9] to be

$$ReZG_3(G) = \sum_{uv \in E(G)} [d_u \, . \, d_v][d_u + d_v]. \tag{1.6}$$

The eccentricity [5] of a vertex v which we denote in this papers as $ecc(v)$ is the maximum distance between v and a vertex farthest away from it by taking the shortest path. Path is a sequence of vertices and edges with the vertices been distinct. We refer to [5, 4, 2, 7] for more explanations on terminologies in graph theory. Eccentricity is also easily obtained from graphs.
There are polynomials whose coefficient or power is a topological index; like the eccentric connectivity polynomial, Zagreb polynomials, forgotten polynomial,

Hosoya polynomial. The eccentricity connectivity polynomial of a graph G is computed as seen in [10] using the formula

$$ECP(G;x) = \sum_{v \in V(G)} d_v x^{ecc(v)} \ . \tag{1.7}$$

As for the Zagreb polynomials the formulae are stated in [1] by Fath-Tabar et al, with the first Zagreb polynomial as

$$M_1(G;x) = \sum_{uv \in E(G)} x^{d_u + d_v} \tag{1.8}$$

and second as

$$M_2(G;x) = \sum_{uv \in E(G)} x^{d_u \ \cdot \ d_v}. \tag{1.9}$$

The Wiener's polynomial [13] which is also called Hosoya polynomial, named after Haruo Hosoya who introduced the polynomial is computed using

$$W(G;x) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} x^{d(u,v)}. \tag{1.10}$$

In this paper we investigated the polynomials of special graphs like the cycle graph $C_n, n \geq 3$, wheel W_n, path P_n, complete graph K_n and complete bipartite graph $K_{m,n}$.The formulae for computing the polynomials are stated.

2. Results

There are some existing results on topological indices of special graphs like the work of Khalifeh et al. [8] where the Zagreb indices of a complete graph K_n, path P_n and a cycle C_n are stated; $M_1(K_n) = n(n-1)^2,\ M_2(K_n) = \frac{n(n-1)^3}{2},\ M_1(P_1) = 0,\ M_1(P_n) = 4n-6, n > 1,\ M_2(P_1) = 0,\ M_2(P_n) = 4(n-2), n > 1,\ M_1(C_n) = 4n, n \geq 3$.

2.1. **Polynomials for a Cycle Graph:** A cycle $C_n, n \geq 3$ is a closed path with distinct vertices.

Remark. For every vertex $v_i \in V(C_n)$, $d_{v_i} = 2$, [2].

Below is a theorem showing the condition needed to make a cycle a closed path.

Theorem 2.1. *A graph C_n is a closed path if and only if its first and second Zagreb indices are equal.*

Proof. ($\Rightarrow$) We assume the graph C_n is a closed path and show that $M_1(C_n) = M_2(C_n)$.
Since C_n is a closed path every vertex $v_i \in V(C_n)$ is adjacent to 2 vertices, v_j and v_k with each on a different path making $d_{(v_i)} = 2$ as seen in the remark on

cycle graph. To get the first Zagreb index we square the degree of each vertex and take the sum for all n vertices, i.e

$M_1(C_n) = \sum_{v_i \in V(C_n)} (d_{v_i})^2 = \sum_{i=1}^{n} (d_{v_i})^2$
$= n(d_{v_i})^2$
$= n\ d_{(v_i)}\ d_{(v_i)}$
$= n\ d_{(v_i)}\ d_{(v_j)}$
$= \sum_{v_i v_j \in E} d_{(v_i)}\ d_{(v_j)} = M_2(C_n) = n.2.2 = 4n.$

($\Leftarrow$) we assume the first and second Zagreb indices are equal and show that C_n is a closed path, i.e the degree of each vertex is 2.

So by assumption $M_1(C_n) = M_2(C_n) = 4n = \sum_{i=1}^{n} (d_{v_i})^2$

$\Rightarrow\ n(d_{v_i})^2 = 4n$
$\Rightarrow\ d_{(v_i)} = 2\ \forall\ v_i \in V(C_n).$ □

Let us look at some theorems showing relationship between topological indices of a closed path C_n.

Theorem 2.2. *The hyper-Zagreb index of a closed path $C_n, n \geq 3$ is four times the first Zagreb index.*

Proof. From Equation 1.5 we have

$$HM(C_n) = \sum_{uv \in E(G)} [d_u + d_v]^2 = \sum_{uv \in E(G)} [2d_v]^2$$

since all vertices of a closed path are of equal degrees. So

$$= \sum_{uv \in E(G)} 4(d_v)^2 = 4 \sum_{uv \in E(G)} (d_v)^2 = 4M_1(C_n).$$

□

Theorem 2.3. *The redefined third Zagreb index of C_n equals its hyper-Zagreb index.*

Proof. The proof is straight forward since $d_{v_i} = 2, \forall\ v_i \in V(C_n)$ implies that $[d_u + d_v] = [d_u\ .\ d_v]$. □

We then consider the polynomials. The eccentricity of a graph is needed in computing its eccentric connectivity polynomial, so we first consider the eccentricity.

Lemma 2.4. *The eccentricity of vertices in a cycle $C_n, n \geq 3$ are equal and given by;*

$$ecc(v) = \begin{cases} \frac{n}{2} & if \;\; n-even, \\ \frac{n-1}{2} & if \; n-odd. \end{cases}$$

Proof. Case 1: For $n - even$.
A cycle $C_n, n \geq 3$ being a closed path has edges between the vertices $V_1, V_2, V_3, ..., V_{\frac{n}{2}}, V_{\frac{n}{2}+1}...V_n$. It is clear that the distance between adjacent vertices V_i and V_{i+1} is; $d(V_i, V_{i+1}) = 1$.

In the sequence of vertices $V_1, V_2, V_3, ..., V_{\frac{n}{2}}, V_{\frac{n}{2}+1}, ..., V_n$. the distance between V_1 and other vertices will be $1, 2, 3, ..., \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, ..., 1$. This is seen from definition, as distances between vertices is obtained by taking the shortest path between the two vertices. So for vertices after $V_{\frac{n}{2}+1}$ the shortest path is gotten by going in the opposite direction. Meaning the distance reaches its peak at the vertex $V_{\frac{n}{2}+1}$ and there are $n - (\frac{n}{2} + 1)$ vertices whose shortest paths are in the opposite direction.

Case 2: For $n - odd$.

In this case there are $n - 1$ vertices excluding the initial vertex V_i, $n - 1$ been even will lead to $\frac{n-1}{2}$ vertices to each side of V_i, i.e for the path in the clockwise and the anticlockwise direction. So the farthest vertex from each vertex V_i is in the position $V_{\frac{n+1}{2}}$ as such the distance between V_i and $V_{\frac{n+1}{2}}$ is going to be $d(V_i, V_{\frac{n+1}{2}}) = \frac{n+1}{2} - 1 = \frac{n-1}{2}$. □

Theorem 2.5. *The eccentric connectivity polynomial of a cycle $C_n, n \geq 3$ is a polynomial of the form*

$$E(C_n, x) = \begin{cases} 2nx^{\frac{n}{2}} & if \;\; n-even, \\ 2nx^{\frac{n-1}{2}} & if \; n-odd. \end{cases}$$

Proof. In a cycle every vertex $v_i \in V(C_n)$ is adjacent to 2 vertices v_{i+1} and v_{i-1}, i.e one before and one after it which implies $d_{(v_i)} = 2$. Using the eccentricity in Lemma 2.4 above and applying the eccentric connectivity polynomial formula in equation 1.7 to sum up for all n vertices gives the result. □

Lemma 2.6. *The Wiener's index of a cycle $C_n, \; n \geq 3$ is given as;*

$$W(C_n) = \begin{cases} \frac{n^3}{8} & for \; n- \;\; even, \\ & \\ \frac{n(n^2-1)}{8} & for \;\; n- \;\; odd. \end{cases}$$

Proof. For n-even.

For each vertex $v_i \in C_n$ the distance is given by $1, 1, 2, 2, ..., \frac{n-2}{2}, \frac{n-2}{2}, \frac{n}{2}$ which means there exist a single vertex v_p such that $d(v_i, v_p) = \frac{n}{2}$. This gives the total distance between vertex v_i and other vertices to be

$$\sum_{i=1}^{\frac{n-2}{2}} 2i + \frac{n}{2}.$$

For the fact that $d(v_i, v_j) = d(v_j, v_i)$ for all n vertices we get;

$$W(C_n) = n \sum_{i=1}^{\frac{n-2}{2}} i + \frac{n^2}{4}.$$

We note that

$$\sum_{i=1}^{\frac{n-2}{2}} i = S_{\frac{n-2}{2}} = \frac{n(n-2)}{8}$$

which gives

$$W(C_n) = \frac{n^3}{8}$$

For n-odd.

For a vertex $v_i \in C_n, i = 1, 2, 3..., n$ there exist two vertices v_j and v_k such that $d(v_i, v_j) = d(v_i, v_k)$ making the distance between v_i and all other vertices to be $1, 1, 2, 2, ..., \frac{n-1}{2}, \frac{n-1}{2}$. This gives the total distance between v_i and all vertices to be

$$\sum_{i=1}^{\frac{n-1}{2}} 2i.$$

Since $d(v_i, v_j) = d(v_j, v_i)$ applying the Wiener's index formula in Equation 1 we have

$$W(C_n) = n \sum_{i=1}^{\frac{n-1}{2}} i$$

but

$$\sum_{i=1}^{\frac{n-1}{2}} i = S_{\frac{n-1}{2}} = \frac{n^2-1}{8}$$

which gives

$$W(C_n) = \frac{n(n^2-1)}{8}.$$

$\square$

Theorem 2.7. *The Wiener's polynomial for a cycle $C_n, n \geq 3$ is given by*

$$W(C_n; x) = \begin{cases} \frac{1}{2}(nx + nx^2 + nx^3 + ... + nx^{\frac{n-1}{2}}) & \textit{for } \ n - \textit{odd}, \\ \\ \frac{1}{2}(nx + nx^2 + nx^3 + ..., + nx^{\frac{n-2}{2}} + nx^{\frac{n}{2}}), & \textit{for } n - \textit{even}. \end{cases}$$

Proof. For n-odd.

Since $d(v_i, v_k) = d(v_k, v_i) \leq \frac{n-1}{2}$, applying the formula for the Wiener's polynomial will give the powers of the indeterminate x ranging from 1 to $\frac{n-1}{2}$ with each coefficient as $n(\textit{the number of edges})$ since the summation runs through pairs of vertices forming the edges.

For n-even.
In this case there is the single vertex v_p as seen in Lemma 2.6 whose distance from each vertex is $\frac{n}{2}$, the powers of the indeterminate will range from 1 to $\frac{n-2}{2}$ and finally $\frac{n}{2}$ which is due to the single vertex. □

2.2. **Polynomials for a wheel W_n.** A wheel [4] $W_n = C_{n-1} + K_1$ is graph derived from a cycle C_{n-1} by adding a single vertex K_1 which is adjacent to all other vertices. The order of a wheel graph is n while the size is given by $2(n-1)$ [2] and $W_3 = K_3$.

Remark. For every vertex $v_i \in W(W_n), d_{v_i} = 3$. if $v_i \in C_{n-1}$ and $d_{v_i} = n - 1$ for the added vertex.

Lemma 2.8. *The hyper-Zagreb index for a wheel is $(n-1)(n^2 + 4n + 40)$.*

Proof. The sum of the degree of pair of vertices from W_n is 6 if both are from C_{n-1} and $3 + (n-1)$ if one is the added vertex K_1, applying the hyper- Zagreb index formula with $|E|$ the size of W_n gives the result;

$$HM(W_n) = \sum_{uv \in E(G)} [d_u \ + \ d_v]^2 = 6^2 \frac{|E|}{2} + [3 + (n-1)]^2 \frac{|E|}{2}$$

$$= 36(n-1) + (n^2 + 4n + 4)(n-1) = (n-1)(n^2 + 4n + 40).$$

□

Theorem 2.9. *The hyper-Zagreb index polynomial of a wheel is given by* $(n - 1)\ x^{(n+2)} + (n-1)x^{36}$.

Proof. Half the total number of edges in a wheel link an added vertex with $(n-1)$ vertices, likewise for the other adjacent vertices which are all of degree 2. This means the coefficient of the hyper-Zagreb polynomial is $\frac{|E|}{2}$ since the power of the indeterminate is the square of the sum of degrees of adjacent vertices which is obtained as 6^2 and $(n+2)$. □

Lemma 2.10. *The redefined third Zagreb index for a wheel is given by* $(n-1)(3n^2+3n+48)$.

Proof. As seen in the proof of Lemma 2.8 the sum of degrees of pair of vertices from W_n is 6, note that the product is 9 if both vertices are from C_{n-1} while we get a sum of $3+(n-1)$ and product of $3(n-1)$ if one vertex is the added vertex K_1. Applying this in Equation 1.6 leads to

$$ReZG_3(W_n) = \sum_{uv\in E(G)} [d_u \ + \ d_v][d_u \ . \ d_v] = 6 \times 9\frac{|E|}{2} + (3+(n-1))(3(n-1))\frac{|E|}{2}$$

$= 54(n-1)+(n+2)(3n-3)(n-1) = (n-1)(3n^2+3n+48) = (n-1)(3n^2+3n+48)$. ☐

Lemma 2.11. *The Wiener's index for a wheel graph is given by* $(n-1)(n-2)$.

Proof. As seen in the remark on a wheel graph, we can see that a vertex $v_i \in C_{n-1}$ is adjacent to 3 vertices making it not adjacent to $(n-4)$ vertices in W_n including itself. This implies that the distance between a vertex v_i and all other vertices is $3+(n-4)\times 2 = (2n-5)$ since the distance between v_i and non-adjacent vertices is 2 and the distance between the added vertex K_1 and other vertices is $(n-1)$. So

$$W(W_n) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(W_n)} d(u,v) = \frac{(2n-5)(n-1)+(n-1)}{2} = (n-1)(n-2).$$

☐

Theorem 2.12. *The Wiener's polynomial for a wheel is given by* $\frac{1}{2}(n-1)\ x^{(2n-5)} + \frac{1}{2}x^{(n-1)}$.

Proof. From Equation 1.10 we can see that the powers of the polynomial is the total distance between a vertex and other vertices in the graph (W_n), making the coefficients to be the number of vertices with equal total distance from other vertices. As shown in the proof of Lemma 2.11 the total distance between the added vertex K_1 and other vertices is $(n-1)$ and $(2n-5)$ for the remaining $(n-1)$ vertices. ☐

Lemma 2.13. *The maximum eccentricity for a wheel graph* (ecc_{max}) *is 2.*

Proof. The added vertex in a wheel graph is adjacent to all vertices of C_{n-1}; making the distance between it and other vertices to be 1 while the distance from a vertex $v_i \in C_{n-1}$ to the farthest vertex say v_j entails passing through added vertex K_1 making the distance $d(v_i, v_j) = 2$. ☐

Theorem 2.14. *The eccentric connectivity polynomial for a wheel graph is a binomial of degree 2.*

Proof. Considering the degree of vertices as shown in remark on a wheel graph and eccentricity of vertices in Lemma 2.13 we use Equation 1.7 to get

$$ECP(W_n; x) = \sum_{v \in V(G)} d_v x^{ecc(v)} = \sum_{v \in V(W_n)} d_{v_i} x^2 + d_{K_1}\ x = 3(n-1)x^2 + (n-1)x.$$

□

2.3. **Polynomials for a path graph P_n.** A path graph P_n is of order n and size $(n-1)$ [5].

Remark. A path graph P_n has 2 edges that link an end vertex of degree 1 to vertices of degree 2 while the remaining $(n-3)$ edges link vertices whose degrees are 2 each [5].

Lemma 2.15. *The hyper-Zagreb index for a path graph P_n is given by* $2(8n-15)$.

Proof. As seen in the remark on a path graph, we could pair vertices of degree 2 in $(n-3)$ ways and the end-vertices with vertices of degree 2 in 2 ways, note that these are vertices which forms an edge in the graph. Applying this to Equation 1.5 we have

$$HM(P_n) = \sum_{uv \in E(G)} [d_u + d_v]^2 = 4^2(n-3) + 2(3)^2 = 2(8n-15).$$

□

Theorem 2.16. *The hyper-Zagreb polynomial for a path graph is given by* $(n-3)x^{16} + 2x^9$.

Proof. The powers of the hyper-Zagreb polynomial is the square of sum of degrees of adjacent vertices and from the proof of Lemma 2.15 we could get $(n-3)$ pairs of vertices each of degree 2 and two pairs of end-vertices with each vertex of degree 2. Applying hyper-Zagreb formula gives

$$HM(G, P_n) = \sum_{uv \in E(G)} x^{[d_u+d_v]^2} = x^{4^2} \times (n-3) + 2x^{3^2} = (n-3)x^{16} + 2x^9.$$

□

Lemma 2.17. *The eccentricity of most vertices of a path graph occur in pairs given by* $ecc(v_i) = ecc(v_j) = (n-i)$ *where* $i+j = n+1$ *and* $i = \begin{cases} 1, 2, 3, ..., \frac{n}{2} & for\ \ n-even, \\ \\ 1, 2, 3, ..., \frac{n-1}{2} & for\ n-odd. \end{cases}$

Proof. for $n - even$.
For a path graph with vertex set $v_1, v_2, v_3, ..., v_n$, observe that $ecc(v_1) = ecc(v_n) = (n-1)$ as they are the farthest vertices away from each other with $d(v_1, v_n) =$

$(n-1)$. Note that the vertex v_n, is the farthest vertex from v_2 leading to the distance $d(v_n, v_2) = (n-2)$, just like v_1 is the farthest from v_{n-1}, leading to $d(v_{n-1}, v_1) = (n-2)$. This implies that $ecc(v_2) = ecc(v_{n-1}) = (n-2)$.
This pattern gives the eccentricity of all vertices of the vertex set of a path graph up to the pair of vertices with $ecc(v_i) = ecc(v_j) = \frac{n}{2}$ which occur at a point where we have $n - n/2$.
For $n - odd$.
Since $v_i, v_j \in V(P_n)$ have equal eccentricity, removing all pair of vertices leaves a single vertex v_p as n is odd. The vertex v_p is at a position $p = \frac{n+1}{2}$ with the equal number $\frac{n-1}{2}$ of vertices to both sides with end vertices as farthest from v_p making $d(v_p, v_1) = d(v_{\frac{n+1}{2}}, v_1) = \frac{n-1}{2}$. □

Theorem 2.18. *The eccentric connectivity polynomial for a path graph is given by* $ECP(P_n; x) =$

$$\begin{cases} 2x^{n-1} + 4x^{n-2} + 4x^{n-3} + ... + 4x^{\frac{n}{2}} & \text{for } n - even, \\ \\ 2x^{n-1} + 4x^{n-2} + 4x^{n-3} + ... + 4x^{\frac{n+1}{2}} + 2x^{\frac{n-1}{2}} & \text{for } n - odd. \end{cases}$$

Proof. As seen in the remark on a path graph that the end vertices are of degree 1 and from Lemma 2.17 it is clear to see that the two have equal eccentricity, similarly for other vertices of degree 2 we have a pair with equal eccentricity except for n odd where we have a single vertex v_p of different eccentricity. We apply the eccentric connectivity polynomial formula;

for $n - even$.
We have $n/2$ values of eccentricity for all vertices since pair of vertices have equal eccentricity making the summation to be;

$$ECP(P_n; x) = \sum_{v \in V(G)} d_v x^{ecc(v)}$$

$$= \sum_{i=1}^{\frac{n}{2}} d_{v_i} x^{(n-i)} = 2(x^{n-1} + 2x^{n-2} + 2x^{n-3} + ... + 2x^{n-\frac{n}{2}})$$

$$= 2(x^{n-1} + 2x^{n-2} + 2x^{n-3} + ... + 2x^{n-\frac{n}{2}})$$

$$= 2x^{n-1} + 4x^{n-2} + 4x^{n-3} + ... + 4x^{\frac{n}{2}}.$$

for $n - odd$.
In this case we have $\frac{n-1}{2}$ equal values of eccentricity for pair of vertices with a single vertex having a different eccentricity, making the summation to be

$$ECP(P_n; x) = \sum_{v \in V(G)} d_v x^{ecc(v)}$$

$$= \sum_{i=1}^{\frac{n-1}{2}} d_{v_i} x^{(n-i)} + d_{v_p} x^{ecc(v_p)} = 2(x^{n-1} + 2x^{n-2} + 2x^{n-3} + ... + 2x^{\frac{n+1}{2}}) + 2x^{\frac{n-1}{2}}$$

$$= 2x^{n-1} + 4x^{n-2} + 4x^{n-3} + ... + 4x^{\frac{n+1}{2}} + 2x^{\frac{n-1}{2}}.$$

□

2.4. **Polynomials for Complete Graph K_n.** A complete graph K_n is of order n and size $\frac{n(n-1)}{2}$ [5].

Remark. All vertices of K_n are adjacent to each other making $d(v_i) = (n-1), \forall\ v_i \in V(K_n)$ [2].

We first look at the hyper-Zagreb index and its polynomial.

Lemma 2.19. *The hyper-Zagreb index for a complete graph K_n is given by $2n(n-1)^3$.*

Proof. Since the degree of each vertex in K_n is $(n-1)$ we sum up the degrees for pair of vertices over the whole edges ,so

$$HM(K_n) = \sum_{uv \in E(G)} [d_u + d_v]^2$$

$$= \sum_{uv \in E(G)} [2(n-1)]^2 = \frac{n(n-1)}{2} \times [2(n-1)]^2 = 2n(n-1)^3.$$

□

Theorem 2.20. *The hyper-Zagreb polynomial for a complete graph is a monomial given by $\frac{n(n-1)}{2} x^{4(n-1)^2}$.*

Proof. The proof of Lemma 2.19 gave the power of the monomial as $[d_u + d_v]^2 = 4(n-1)^2$ which is similar for all pair of vertices forming an edge in a complete graph which means the coefficient is the size of K_n. □

Theorem 2.21. *The Wiener's polynomial for a complete graph K_n is a monomial of degree 1.*

Proof. The power of an indeterminate in a Wiener's polynomial is the distance between every pair of vertices in a graph, in the case of a complete graph all vertices are adjacent making $d(u, v) = 1, \forall\ u, v \in K_n$ and the coefficient is the size of K_n since $d(v_i, v_j) = d(v_j, v_i)$. □

We now take a look at the eccentric connectivity polynomial for a complete graph.

Theorem 2.22. *The eccentric connectivity polynomial for a complete graph K_n is a monomial of degree 1 given by $n(n-1)x$.*

Proof. Since the vertices of a complete graph are all adjacent it implies that the eccentricity of every vertex in K_n is 1 and the degree of each vertex is $(n-1)$ as seen in the remark on a complete graph. Applying the eccentric connectivity polynomial gives

$$ECP(K_n;x) = \sum_{v\in V(K_n)} d(v)x^{ecc(v)} = \sum_{i=1}^{n} d(v_i)x^{ecc(v_i)} = n(n-1)x.$$

□

2.5. **Polynomials for a complete Bipartite Graph $K_{m,n}$.** A complete bipartite graph $K_{m,n}$ is a graph whose vertex set is partitioned into two, it is of order $m+n$ and size mn [5].

Remark. If the vertex set of a complete graph is partitioned into two sets say A and B with $|A| = m$ and $|B| = n$, then for $v_i \in A$ and $v_j \in B$ we have $d(v_i) = n$, $d(v_j) = m$ [5].

Lemma 2.23. *The hyper-Zagreb index for $K_{m,n}$ is given by $mn(m^2 + n^2)$.*

Proof. The degree of vertices is given in the remark on a complete bipartite graph, getting the square of the sum of degrees for two vertices which forms an edge and adding across all edges results to

$$HM(K_{m,n}) = \sum_{uv\in E(K_{m,n})} [d_v + d_v]^2 = mn(m+n)^2.$$

□

Theorem 2.24. *The hyper-Zagreb polynomial for a complete bipartite graph is a monomial given by $mn\ x^{(m+n)^2}$.*

Proof. It is a monomial with power as the square of the sum of degrees of two adjacent vertices which is clearly shown as $(m+n)^2$ for a single edge and since the edges have equal sum of degrees for its pair of vertices; the size of the graph represents the coefficient. □

Theorem 2.25. *The Wiener's polynomial of a complete bipartite graph is given by; $W(K_{m,n};x) = mnx + \frac{[m(m-1)+n(n-1)]x^2}{2}$.*

Proof. The polynomial has the power of the indeterminate as the distance from a vertex and every other vertices in the graph with coefficients as number of vertices with equal distances. The vertex set of the graph has partitions say A and B with $|A| = m$ and $|B| = n$; $\forall\ u \in A$ and $\forall\ v \in B$, $d(u,v) = 1$ since $u \in A$ is adjacent to every $v \in B$. And $u, v\ \in A\ \ (or\ B)\ d(u,v) = 2$ since there is a vertex $w \in B$ $(or\ A)$ between them, so a single vertex from partition A will lead to $nx+(m-1)x^2$ terms of the polynomial while for all vertices from A the terms will be

(i) $$mnx+m(m-1)x^2.$$

In a similar way, a vertex from B will lead to $mnx+(n-1)x^2$ terms of the polynomial, for all vertices from B we get

(ii) $$mnx+n(n-1)x^2$$

since $d(u.v)=d(v,u)$.

$$W(G;x)=mnx+\frac{[m(m-1)+n(n-1)]x^2}{2}.$$

□

Conclusion. The study derives formulae for computing polynomials associated with topological indices of special graphs such as the cycle, wheel, path, complete graph, and complete bipartite graph. It is shown that the hyper-Zagreb index of a closed path C_n is four times its first Zagreb index, and that it coincides with the redefined Zagreb index for the same graph. Furthermore, a cycle graph is shown to be a closed path if and only if its first and second Zagreb indices are equal.

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References

[1] Astaneh-Asl A. & Fath-Tabar G. H. (2011), Computing the First and Third Zagreb Polynomials of Cartesian Products of Graphs. *Iranian J. Math. Chem.* **2(2)**: 73-78.
[2] Balakrishnan V. K. (1997). Theory and Problems of Graph Theory. *McGraw-Hill Companies* U.S.A.
[3] Basavanagoud B., Desal V. R., Mirajkar K. G., Pooja B. & Caugul I. N. (2020), Four New Tensor Productof graphs and their Zagreb indices and coindices. *Electronic Journal of Mathematical Analysis and Applications* **8(1)**: 209-219.
[4] Bondy J. A. & Murty U. S. R. (1982). Graph Theory With Applications. 5th Edition, *New York, Elsevier Science Publishing Co. Inc.*
[5] Chartrand G. & Zhang P. (2012). *A First Course in Graph Theory.* Dover Publications. Inc. Mincola, New York.
[6] GutmaN I. & Trinajstic N. (1972), Graph Theaory and molecular orbitals. Total $\pi-$ electron energy of alternant hydrocarbons. *Chem. Phys. Lett.*, **17(4)**: 535-538.

[7] Harary F. (1969). Graph Theory. *Addison-Wesley*, Reading.

[8] khalifeh M. H., Youseffi-Azari H. & Ashrafi A. R. (2009). The first and Second Zagreb indices of some graph operations. *Discrete Applied Mathematics* **157(4)**: 804-811.

[9] Parvez A., Kirmani S. A. K., Al-Rugaie O. & Azam F. (2020). Degree Based Topological Indices and Polynomials of hyaluronic acid-corcumin Conjugates. *Saudi Pharmaceutical Journal* **28(9)**: 1093-1100.

[10] Rajarethinam D., Baig A. Q., Sajjad W. & Farahani M. R. (2017). Eccentric Connectivity Polynomial and Total Eccentricity Polynomial of NA_m^n Nanotube. *Journal of Informatics and Mathematical Sciences* **9(1)**: 201-215.

[11] Rao V., Kanwal A., Abbas R., Noureen S., Fahad A. & Qureshi M. I. (2021). Some Degree Based Topological Indices of Carboxy-terminated dendritic macromolecules. *Main Group Met. Chem.***44**: 165-172.

[12] Shirdel G. H., Rezapour H. & Sayadi A. M. (2013). General Fifth-M-Zagreb Indices and Fifth-M-Zagreb Polynomials of PAMAM Dendrimers. *International Journal of Fuzzy Mathematical Archives* **4(2)**: 213-220.

[13] Wiener H. (1947). Structural determination of the paraffin boiling points. *J. Am. Chem. Soc.***26**: 17-20.