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Flux-Hardy Inequalities with Optimal Constants via Divergence and Rearrangements

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Abstract

We develop sharp Hardy-type inequalities for boundary flux functionals generated by dilations of a fixed smooth set in $\mathbb{R}^n$. The method combines the divergence theorem with rearrangement bounds to reduce flux estimates to one-dimensional Hardy averages. Directional, divergence-form, and multi-flux inequalities are obtained in arbitrary dimension with the optimal constant $\left(\frac{p}{p-1}\right)^p$, and sharpness is shown by explicit near-extremizing constructions.

1. Introduction and Preliminaries

Hardy-type inequalities occupy a central place in analysis because they encode a robust principle: a suitable average or cumulative quantity is controlled in L^p by a derivative-type quantity with an explicit constant. In one dimension, the Hardy operator

$$(Hh)(t) = \frac{1}{t}\int_0^t h(s)\,ds \tag{1}$$

is a canonical averaging operator, and Hardy's inequality identifies its boundedness on $L^p(0,\infty)$ along with the exact best constant [1, 4]. A classical theme

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is that sharpness questions are often resolved not by a single extremizer but by sequences that nearly saturate an inequality. Rearrangement theory provides a systematic language for such constructions and for reducing multidimensional integral inequalities to one-dimensional monotone problems [4, 5].
Recent progress has shown that Hardy's inequality admits refinements that preserve the sharp constant while strengthening the left-hand side. A notable example is the improved one-dimensional Hardy inequality of Frank–Laptev–Weidl, where the classical Hardy quantity is replaced by a two-sided maximal functional capturing both local and far-field behavior, yet the constant remains $\frac{p}{p-1}$ [1]. Motivations for such refinements arise naturally in spectral estimates for Schrödinger operators and in the broader landscape of sharp functional inequalities [2].
A complementary geometric approach to Hardy-type estimates proceeds via flux identities. The divergence theorem converts boundary flux integrals into volume integrals, and rearrangement bounds then dominate volume integrals over sets of fixed measure by one-dimensional integrals of decreasing rearrangements. This divergence–rearrangement route has been used recently to obtain sharp Hardy-type inequalities for boundary flux functionals in three dimensions, with sharpness established by explicit near-extremizers of the Hardy operator together with a realization principle for rearrangements on dilation families [5]. Multidimensional improvements related to the Frank–Laptev–Weidl phenomenon have also been pursued via radialisation and symmetric decreasing rearrangements [3].
The present work develops a unified framework that yields sharp multidimensional Hardy-type inequalities for boundary flux functionals generated by dilations of a fixed smooth set. The main results include a directional flux inequality for scalar potentials, a divergence-form inequality for vector fields, and a multi-flux inequality controlled by $\|\nabla f\|_{L^p}$, each with the sharp constant $\left(\frac{p}{p-1}\right)^p$ inherited from the one-dimensional Hardy operator [1].

2. Preliminaries

2.1. **Notation.** Throughout the paper, $n \geq 2$ and $1 < p < \infty$. We write μ for Lebesgue measure on $\mathbb{R}^n$, and for measurable $E \subset \mathbb{R}^n$ we set $|E| := \mu(E)$. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm and $x \cdot y$ the Euclidean inner product. The indicator of a set E is denoted by $\mathbf{1}_E$ [4].
Given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, $\|u\|_{L^p(\mathbb{R}^n)}$ denotes its L^p-norm, and $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ means that u has weak first derivatives locally in L^p. For vector fields $V : \mathbb{R}^n \to \mathbb{R}^n$, $\operatorname{div} V$ denotes the distributional divergence and ∇u the weak gradient [2].
A fixed reference set $D \subset \mathbb{R}^n$ is assumed open with smooth boundary, $0 \in D$, and $|D| = 1$. For $t > 0$, $D_t = t^{1/n} D$ denotes the dilation so that $|D_t| = t$. The outward unit normal on ∂D_t is ν, and dS denotes surface measure on ∂D_t [5].

Constants denoted by C may change from line to line but depend only on the fixed parameters indicated (typically p, n, and D). The notation $A \lesssim B$ means $A \leq CB$ for such a constant C, and $A \simeq B$ means $A \lesssim B \lesssim A$ [4].

2.2. **Rearrangements and distribution functions.** For a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ define its distribution function

$$\mu_u(\lambda) = \mu(\{x \in \mathbb{R}^n : \ |u(x)| > \lambda\}), \qquad \lambda \geq 0, \tag{2}$$

and its non-increasing rearrangement

$$u^*(t) = \inf\{\lambda > 0 : \ \mu_u(\lambda) \leq t\}, \qquad t > 0, \tag{3}$$

as in the divergence–Hardy approach to flux inequalities [5]. Define also

$$u^{**}(t) = \frac{1}{t} \int_0^t u^*(s)\, ds \tag{4}$$

A basic rearrangement domination used in the flux setting is: for any measurable $E \subset \mathbb{R}^n$ with $\mu(E) = t$ and $u \geq 0$,

$$\int_E u(x)\, dx \leq \int_0^t u^*(s)\, ds = t\, u^{**}(t) \tag{5}$$

which is the key reduction step used in [5].

2.3. **Dilation families of fixed volume.** Fix an open set $D \subset \mathbb{R}^n$ such that $0 \in D$, ∂D is smooth, and $\mu(D) = 1$, and define

$$D_t = t^{1/n} D, \qquad t > 0,$$

so that $\mu(D_t) = t$. This normalization is used for sharp flux inequalities in [5].

2.4. **Hardy's inequality and improved maximal structure.** For $1 < p < \infty$, the Hardy operator satisfies

$$\|Hh\|_{L^p(0,\infty)} \leq \frac{p}{p-1} \|h\|_{L^p(0,\infty)}, \qquad (Hh)(t) = \frac{1}{t} \int_0^t h(s)\, ds,$$

with sharp constant; see [1, 4]. Moreover, a refined inequality controls a two-sided maximal Hardy expression with the same sharp constant [1].

2.5. **Divergence theorem in the flux setting.** For sufficiently regular vector fields $W : \mathbb{R}^n \to \mathbb{R}^n$, the divergence theorem yields

$$\int_{\partial D_t} W \cdot \nu\, dS = \int_{D_t} \operatorname{div} W\, dx \tag{6}$$

and this identity is the geometric bridge from boundary flux to volume integrals in the divergence–Hardy method [5].

3. Results

3.1. **Setting and basic inequalities.** Let $n \geq 2$ and μ denote Lebesgue measure on $\mathbb{R}^n$. For measurable $u : \mathbb{R}^n \to \mathbb{R}$ define

$$u^*(t) = \inf\{\lambda > 0 : \mu(\{x : |u(x)| > \lambda\}) \leq t\}, \qquad t > 0,$$

and

$$u^{**}(t) = \frac{1}{t} \int_0^t u^*(s)\, ds.$$

Fix an open set $D \subset \mathbb{R}^n$ with $0 \in D$, smooth boundary ∂D, and $\mu(D) = 1$. Define $D_t = t^{1/n} D$ so that $\mu(D_t) = t$.
We use: if $u \geq 0$ and $\mu(E) = t$, then

$$\int_E u(x)\, dx \leq \int_0^t u^*(s)\, ds = t\, u^{**}(t) \tag{7}$$

Lemma 3.1 (Hardy bound for u^{**})**.** *Let $1 < p < \infty$. Then*

$$\int_0^\infty \big(u^{**}(t)\big)^p\, dt \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \big(u^*(t)\big)^p\, dt \tag{8}$$

and the constant $\left(\frac{p}{p-1}\right)^p$ is sharp.

Proof. Let $h(t) = u^*(t) \geq 0$. Then $u^{**}(t) = (Hh)(t)$ where $(Hh)(t) = \frac{1}{t}\int_0^t h(s)\, ds$. The classical Hardy inequality gives $\|Hh\|_{L^p(0,\infty)} \leq \frac{p}{p-1}\|h\|_{L^p(0,\infty)}$. Raising to the p-th power yields the claim, and sharpness follows from sharpness of the Hardy operator norm. ☐

Lemma 3.2 (Near-extremizers for the Hardy operator)**.** *Let $1 < p < \infty$. For every $\varepsilon > 0$ there exists a non-increasing $\varphi_\varepsilon : (0, \infty) \to [0, \infty)$ such that*

$$\frac{\|H\varphi_\varepsilon\|_{L^p(0,\infty)}}{\|\varphi_\varepsilon\|_{L^p(0,\infty)}} \geq \frac{p}{p-1} - \varepsilon \tag{9}$$

Proof. Fix small $\delta > 0$ and large $R > 0$. Let $\varphi(t) = (t+\delta)^{-1/p}\mathbf{1}_{(0,R)}(t)$. A direct computation shows $(H\varphi)(t)$ is close to $\frac{p}{p-1}\varphi(t)$ on most of $(0, R)$. Choosing δ sufficiently small and R sufficiently large yields the stated ratio. ☐

Lemma 3.3 (Realization on the dilation family)**.** *Let $\varphi : (0, \infty) \to [0, \infty)$ be non-increasing. Then there exists measurable $u : \mathbb{R}^n \to [0, \infty)$ such that $u^*(t) = \varphi(t)$ for a.e. $t > 0$ and*

$$\int_{D_t} u(x)\, dx = \int_0^t \varphi(s)\, ds \qquad \textit{for all } t > 0 \tag{10}$$

Proof. Approximate φ from above by decreasing step functions and define u_m to be constant on annuli $D_{t_k} \setminus D_{t_{k-1}}$ with heights matching the steps; then pass to the limit. ☐

3.2. **Directional flux inequality.** Fix $a \in \mathbb{R}^n$. For $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ define

$$\mathcal{F}_a(t) = \int_{\partial D_t} f(x)\, a \cdot \nu(x)\, dS(x) \tag{11}$$

where ν is the outward unit normal.

Theorem 3.4 (Directional flux inequality; sharp)**.** *Let $n \geq 2$ and $1 < p < \infty$. Then for all $a \in \mathbb{R}^n$ and $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\int_0^\infty \max\left\{ \sup_{0<s\leq r} \frac{|\mathcal{F}_a(s)|^p}{r^p},\ \sup_{r<s<\infty} \frac{|\mathcal{F}_a(s)|^p}{s^p} \right\} dr \leq \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^n} |a \cdot \nabla f(x)|^p\, dx \tag{12}$$

and the constant is sharp.

Proof. Apply the divergence theorem to the vector field fa. Since a is constant, $\nabla \cdot (fa) = a \cdot \nabla f$, hence

$$\mathcal{F}_a(t) = \int_{\partial D_t} (fa) \cdot \nu\, dS = \int_{D_t} a \cdot \nabla f\, dx \tag{13}$$

Let $u(x) = |a \cdot \nabla f(x)|$. Then

$$|\mathcal{F}_a(t)| \leq \int_{D_t} u \leq \int_0^t u^*(s)\, ds = t\, u^{**}(t) \tag{14}$$

Because u^{**} is non-increasing, for $r > 0$,

$$\sup_{0<s\leq r} \frac{|\mathcal{F}_a(s)|}{r} \leq u^{**}(r), \qquad \sup_{r<s<\infty} \frac{|\mathcal{F}_a(s)|}{s} \leq u^{**}(r).$$

Thus the integrand is bounded by $(u^{**}(r))^p$ and

$$\int_0^\infty (\cdots)\, dr \leq \int_0^\infty (u^{**}(r))^p\, dr \tag{15}$$

Apply Lemma 3.1 and use $\|u^*\|_{L^p(0,\infty)} = \|u\|_{L^p(\mathbb{R}^n)}$.
For sharpness, take φ_ε from Lemma 3.2 and realize it by u_ε from Lemma 3.3. Choose coordinates so $a = |a|e_1$ and define

$$f_\varepsilon(x_1, x') = \frac{1}{|a|} \int_0^{x_1} u_\varepsilon(s, x')\, ds,$$

so that $a \cdot \nabla f_\varepsilon = u_\varepsilon$. Then

$$\frac{|\mathcal{F}_a(t)|}{t} = \frac{1}{t} \left| \int_{D_t} u_\varepsilon \right| = \frac{1}{t} \int_0^t \varphi_\varepsilon(s)\, ds = (H\varphi_\varepsilon)(t) \tag{16}$$

and the ratio approaches $\left(\frac{p}{p-1}\right)^p$ as $\varepsilon \downarrow 0$. ∎

Corollary 3.5 (One-sided forms; sharp)**.** *Under the hypotheses of Theorem 3.4,*

$$\int_0^\infty \left(\sup_{0<s\le r} \frac{|\mathcal{F}_a(s)|}{r} \right)^p dr \le \left(\frac{p}{p-1} \right)^p \int_{\mathbb{R}^n} |a \cdot \nabla f|^p \, dx,$$

and

$$\int_0^\infty \left(\sup_{r<s<\infty} \frac{|\mathcal{F}_a(s)|}{s} \right)^p dr \le \left(\frac{p}{p-1} \right)^p \int_{\mathbb{R}^n} |a \cdot \nabla f|^p \, dx,$$

with best constant.

Proof. Drop one of the two terms inside the maximum in Theorem 3.4. Sharpness follows from sharpness of Theorem 3.4. □

3.3. **Vector-field divergence inequality.** Let $V \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ and define

$$\Phi_V(t) = \int_{\partial D_t} V(x) \cdot \nu(x) \, dS(x) \tag{17}$$

Theorem 3.6 (Vector-field flux inequality; sharp)**.** *Let $n \ge 2$, $1 < p < \infty$, and assume* $\operatorname{div} V \in L^p(\mathbb{R}^n)$*. Then*

$$\int_0^\infty \max \left\{ \sup_{0<s\le r} \frac{|\Phi_V(s)|^p}{r^p}, \sup_{r<s<\infty} \frac{|\Phi_V(s)|^p}{s^p} \right\} dr \le \left(\frac{p}{p-1} \right)^p \int_{\mathbb{R}^n} |\operatorname{div} V(x)|^p \, dx \tag{18}$$

and the constant is sharp.

Proof. By the divergence theorem,

$$\Phi_V(t) = \int_{D_t} \operatorname{div} V \, dx.$$

Let $u = |\operatorname{div} V|$. Then $|\Phi_V(t)| \le \int_{D_t} u \le t\, u^{**}(t)$. Proceed as in Theorem 3.4 to bound the maximal functional by $(u^{**}(r))^p$, integrate, and apply Lemma 3.1. Sharpness follows by realizing near-extremizers for u and constructing a vector field with that divergence. □

3.4. **Multi-flux inequality.** Let $a_1, \ldots, a_m$ be orthonormal and define

$$\mathcal{F}_{a_j}(t) = \int_{\partial D_t} f(x)\, a_j \cdot \nu(x) \, dS(x) \tag{19}$$

Theorem 3.7 (Multi-flux inequality; sharp)**.** *Let $n \ge 2$, $1 < p < \infty$, and $a_1, \ldots, a_m$ be orthonormal. Then for all $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\int_0^\infty \left(\sum_{j=1}^m |\mathcal{F}_{a_j}(t)|^2 \right)^{p/2} \frac{dt}{t^p} \le \left(\frac{p}{p-1} \right)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx \tag{20}$$

and the constant is sharp.

Proof. For each $t > 0$, by ℓ^2 duality,

$$\left(\sum_{j=1}^{m} |\mathcal{F}_{a_j}(t)|^2\right)^{1/2} = \sup_{|\beta|=1} \left|\int_{\partial D_t} f(x)\, a_\beta \cdot \nu \, dS\right|, \quad a_\beta = \sum_{j=1}^{m} \beta_j a_j, \ |a_\beta| = 1.$$

By the divergence theorem,

$$\int_{\partial D_t} f\, a_\beta \cdot \nu \, dS = \int_{D_t} a_\beta \cdot \nabla f \, dx.$$

Hence

$$\left|\int_{D_t} a_\beta \cdot \nabla f \, dx\right| \le \int_{D_t} |\nabla f| \, dx \le \int_0^t (|\nabla f|)^*(s) \, ds \tag{21}$$

Therefore,

$$\left(\sum_{j=1}^{m} |\mathcal{F}_{a_j}(t)|^2\right)^{1/2} \le \int_0^t (|\nabla f|)^*(s) \, ds \tag{22}$$

Divide by t, raise to the p-th power, integrate in t, and apply Lemma 3.1 with $u = |\nabla f|$. Sharpness follows by restriction to $m = 1$ and Theorem 3.4. $\square$

Conclusion. The divergence–rearrangement method yields a direct and flexible route from one-dimensional Hardy averages to sharp multidimensional flux inequalities along dilation families. Boundary fluxes are converted to volume integrals by divergence identities and then dominated by Hardy averages of rearrangements. This produces directional, divergence-form, and multi-flux inequalities with the optimal constant $\left(\frac{p}{p-1}\right)^p$. Sharpness follows from explicit near-extremizers for the Hardy operator together with a realization principle that matches rearranged profiles to functions supported on the dilation family.

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