



Fixed Point with Weakly Compatible Function on Semigroup of Linear Operator

AKINYELE AKINOLA YUSSUFF*, MICHAEL OYELAMI AJISOPE, OBASHOLA
RAFIU ZUBAIR, IDAYAT FOLUKE USAMOT AND KAMILU RAUF

ABSTRACT

This paper consists of fixed point results on ω -order preserving partial contraction mapping as a weakly compatible function. Moreover, the results are established as semigroup of linear operator.

1. INTRODUCTION

Let X be a Banach space, $X_n \subseteq X$ be a finite set and $S \subseteq X_n$ be subspace of X , $L(x)$ be a bounded linear operator in X . Suppose $(T(t))_{t \geq 0}$ is C_0 -semigroup, $\omega - OCP_n$ is ω -order preserving partial contraction mapping and $M_m(N)$ is a matrix.

Fixed point theory is concern with finding the condition to the structure that a set X must be endowed and as well as the properties of operator $T : X \rightarrow X$ in order to obtain results on existence, uniqueness and construction of a fixed point [2]. Riesz space value is introduced as our motivation for this paper where Riesz space is the norm.

A Riesz space is an ordered vector space and a lattice. Let $\mathcal{J}$ be a Riesz space with positive cone $\mathcal{J}_+ = \{x \in X : x \geq 0\}$. If (α_n) is a decreasing sequence in $\mathcal{J}$ such that $\inf \alpha_n = \alpha$, we write $\alpha_n \rightarrow \alpha$. This paper will focus on fixed point result

Received: 29/09/2019, Accepted: 12/11/2019, Revised: 29/11/2019. * Corresponding author.
2015 *Mathematics Subject Classification*. 47H10, & 47S10.

Key words and phrases. ω -order preserving partial contraction mapping, Banach space,
Bounded linear operator, C_0 -Semigroup, Archimedean

Department of Mathematics, University of Ilorin, P. M. B. 1515 Ilorin, Kwara State, Nigeria

Department of Mathematics, Federal University Oye, PMB 373, Ekiti State, Nigeria

E-mail: olaakinyele04@gmail.com

on $\omega - OCP_n$ as a weakly compatible function of semigroup of linear operator. Altum *et al.* [1], obtained fixed points of weakly compatible maps satisfying a general contractive condition of integral type. Abbas and Jumyeh [2], established common fixed point results for non-commuting mappings without continuity in cone metric spaces. Alata *et al.* [3], proved some results on common fixed point for generalized f-contraction Mapping. Vrabie [5] deduced C_0 -Semigroup and Application, & Rauf and Akinyele [6], generated properties of ω -order preserving partial contraction mapping and its relation to C_0 -semigroup. Many authors including Rauf and Yusuf [7] worked on common fixed point theorems in vector metric spaces.

2. PRELIMINARIES

Definition 2.1 (Banach space)

A Banach space is a norm space which is complete in the sense that every Cauchy sequence converges to a particular point on the space.

Definition 2.2 (Archimedean)

The Riesz space $\mathcal{J}$ is said to be Archimedean if $\frac{1}{n}\alpha \rightarrow 0$ holds for every $\alpha \in \mathcal{J}_+$

Definition 2.3 (Semigroup)

A semigroup is a pair $(S, *)$ in which S is a non empty set and $*$ is a binary associative operation on S , that is, the equation $(x * y) * z = x * (y * z)$ holds for all $x, y, z \in S$.

Definition 2.4 (C_0 -semigroup)

A C_0 -semigroup is a strongly continuous one parameter semigroup of bounded linear operator on a Banach space.

Definition 2.5 ($\omega - OCP_n$)

A transformation $\alpha \in S$ is called ω -order preserving partial contraction mapping if $\forall x, y \in Dom \alpha : x \leq y \Rightarrow \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t + s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise $T(0) = I$.

Definition 2.6 (Infinitesimal generator)

Let $T(t)$ be a C_0 -semigroup, the infinitesimal generator of $T(t)$ (denoted by A) is given by

$$\lim_{t \rightarrow 0} A_t x = \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} \forall x \in X$$

Example 1

Example of semigroup of linear operator is $\omega - OCP_n$ which is a subset of C_0 -semigroup and the simplest one among them is 2×2 matrix.

Suppose $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $T(t) = e^{tA}$, then $e^{tA} = \begin{pmatrix} e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}$

Example 2

Example of 3×3 matrix in the same operator is $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ - & 2 & 3 \end{pmatrix}$ and $T(t) =$

$$e^{tA}, \text{ then } e^{tA} = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & e^{2t} & e^{2t} \\ I & e^{2t} & e^{3t} \end{pmatrix}$$

Example 3

It is well known that $\mathbb{R}^2$ is a Riesz space with coordinate ordering defined by:

$$(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2$$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We can equally have

$$(x_1, y_1, z_1) = (x_2, y_2, z_2) \Leftrightarrow x_1 = x_2, y_1 = y_2 \text{ and } z_1 = z_2$$

for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$ which can be generalized as $(x_1, y_1, z_1, \dots, P_n) = (x_2, y_2, z_2, \dots, P_{n+1}) \Leftrightarrow x_1 = x_2, y_1 = y_2, z_1 = z_2 \text{ and } P_n = P_{n+1}$ and for $(x_1, y_1, z_p, \dots, P_n), (x_2, y_2, z_2, \dots, P_{n+1}) \in \mathbb{R}^n$

3. MAIN RESULTS

In this section, fixed point results on $\omega - OCP_n$ which is considered as a weakly compatible function and also as a semigroup of linear operator are obtained.

Theorem 3.1

Let X be a Banach space with $\mathcal{J}$ Archimedean and $T : X \rightarrow X$ be an $\omega - OCP_n$ (order preserving partial contraction mapping) for which $a \in (0, 1)$ whenever $\|Tx - Ty\| \leq a \|x - y\| \forall x, y \in X$. Then, T has a fixed point $u \in X$. Furthermore, for any given $x \in X$ there exists a limit with

$$(1) \quad \|T^n(x) - u\| \leq \frac{a^n}{1-a} \|x - T(x)\|$$

However, we have $T(X) \subseteq T_1(X)$. If we have another map $T_1 : X \rightarrow X$ which is also $\omega - OCP_n$ and $T_1(X)$ or $T(X)$ is E-complete subspace on X . If T_1 and T are weakly compatible, then they have unique common fixed point on X , where

$$u(x, y) \in \{\|T_1x - T_1y\|, \|T_1x - Tx\|, \|T_1y - Ty\|, \frac{1}{2}[\|T_1x - T_y\|] + \|T_1y - Tx\|\}$$

Proof

Suppose T that maps a complete normed space onto a complete normed space is Archimedean for which $a \in (0, 1)$ and for all $x \in X$ then

$$\|Tx - Ty\| \leq a \|x - y\| \text{ and}$$

$$\|T^n(x) - T^{n+1}(x)\| \leq a \|T^{n-1}(x) - T^n(x)\| \leq \dots \leq a^n \|x - T(x)\|$$

Thus, for $m > n$ where $n \in \mathbb{N} \cup \{0\}$

$$\|T^n(x) - T^m(x)\| \leq \|T^n(x) - T^{n+1}(x)\| + \|T^{n+1}(x) - T^{n+2}(x)\| + \dots +$$

$$\begin{aligned} \|T^{m-1}(x) - T^m(x)\| &\leq a^n \|x - T(x)\| + \cdots + a^{m-1} \|x - T(x)\| \\ &\leq a^n \|x - T(x)\| [1 + a + a^2 + \cdots] = \frac{a^n}{1-a} \|x - T(x)\| \end{aligned}$$

hence

$$(2) \quad \|T^n(x) - T^m(x)\| = \frac{a^n}{1-a} \|x - T(x)\|$$

Suppose there exists $x, y \in X$ with $x = Tx$ and $y = Ty$, then by contraction mapping, we have

$$\|Tx - Ty\| \leq a \|x - y\|$$

and

$$\|Tx - Ty\| \leq \|x - y\|$$

Therefore $\|x - y\| = 0$ then

$$x = y$$

Since X is complete, there exist $u \in X$ such that

$$\lim_{n \rightarrow \infty} T^n(x) = u$$

Moreover, the continuity of T yields

$$u = \lim_{n \rightarrow \infty} T^{n+1}(x) = \lim_{n \rightarrow \infty} T(T^n(x)) = T(u)$$

Therefore u is a fixed point of T . Hence, letting $m \rightarrow \infty$ in (2), then

$$(3) \quad \|T^n(x) - u\| \leq \frac{a^n}{1-a} \|x - T(x)\|$$

Now, we need to show that T_1 and T are weakly compatible and have a unique common fixed point on X . Since $\mathcal{J}$ is Archimedian on X and let $T_1 : X \rightarrow X$ be another self map on X such that the range of T_1 contains the range of T and the range of at least one is E -complete whenever x_n is an E -Cauchy sequence, then there exists $z \in T_1(x)$ such that $T_1 y_n \rightarrow z$. Hence there exists a sequence k_n in $\mathcal{J}$ such that $k_n \rightarrow 0$ and $\|Ty_n - z\| = k_n$. On the other hand, we can find $v \in X$ such that $T_1 v = z$. Let us show that $Tv = z$. Then

$$\|Tv - z\| \leq \|Tv - Ty_n\| + \|Ty_n - z\| \leq au \|y_n - v\| + k_{n+1}$$

For all $n \in \mathbb{N}$,

$$x(y_n, v) \in \{\|T_1 y_n - T_1 v\|, \|T_1 y_n - Ty_n\|, \|T_1 v - Tv\|, \frac{1}{2} [\|T_1 y_n - Tv\| + \|T_1 v - Ty_n\|]\}$$

at least one of the following four hold:

- i. $\|Tv - z\| \leq \|T_1 y_n - T_1 v\| + k_{n+1} \leq k_n + k_{n+1} = 2k_n$;
- ii. $\|Tv - z\| \leq \|T_1 y_n - Ty_n\| + k_{n+1} \leq \|T_1 y_n - z\| + 2k_{n+1} \leq 3k_n$;
- iii. $\|Tv - z\| \leq a \|T_1 v - Tv\| + k_{n+1} \leq a \|Tv - z\| + k_n \Rightarrow \|Tv - z\| \leq \frac{1}{1-a} k_n$;

and

$$\begin{aligned}
\text{iv. } \|Tv - z\| &\leq \frac{1}{2} [\|T_1 y_n - Tv\|] + \|T_1 v - T y_n\| + k_{n+1} \\
&\leq \frac{1}{2} \|T_1 y_n - T v\| + \frac{3}{2} k_{n+1} \\
&\leq \frac{1}{2} \|T_1 y_n - z\| + \frac{1}{2} \|Tv - z\| + \frac{3}{2} k_n \\
&\leq \frac{1}{2} \|Tv - z\| + 2k_n
\end{aligned}$$

That is $\|Tv - z\| \leq 4k_n$. Since the infimum of the sequence on the right side of last inequality is zero, then $\|Tv - z\| = 0$ i.e $Tv = z$. Therefore, z is a point of coincidence of T and T_1 . If z_1 is another point of coincidence then there is $v_1 \in X$ with $z_1 = Tv_1 = T_1 v_1$. Now from our contractive condition it follows that

$$\|z - z_1\| = \|Tv - T v_1\| \leq a \|v - v_1\|$$

where

$$\begin{aligned}
\|v - v_1\| &\in \{\|T_1 v - T_1 v_1\|, \|T_1 v - Tv\|, \|T_1 v_1 - T v_1\|, \frac{1}{2}[\|T_1 v - T v_1\| + \|T_1 v_1 - Tv\|]\} \\
&= \{0, \|z - z_1\|\}
\end{aligned}$$

Hence, $\|z - z_1\| = 0$, implies that $z = z_1$. If T_1 and T are weakly compatible, then it is obvious that z is unique common fixed point of T and T_1 . Hence the proof.

Theorem 3.2

Let $A \in \omega - OCP_n$ such that $\omega - OCP_n \in L(X)$ where $L(X)$ is a bounded linear operator from X to X then $T(t) = e^{tA}$ with $t \in \mathbb{N}$ is uniformly continuous semigroup.

Proof

Since A is bounded, we have that $\|A\| < \infty$ and thus $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ converges for $t \geq 0$ to the bounded linear operator $T(t)$ since $\left(\sum_{i=0}^{\infty} \frac{(t)^i}{i!}\right) \left(\sum_{j=0}^{\infty} \frac{(s)^j}{j!}\right) = \sum_{k=0}^{\infty} \frac{(t+s)^k}{k!}$. $T(0) = I$ is clear, It remains to show that $T(t)$ is a uniformly continuous semigroup, we have

$$\begin{aligned}
\|T(t) - I\| &= \left\| \sum_{k=1}^{\infty} \frac{t^k A^k}{k!} \right\| \\
&\leq \sum_{k=1}^{\infty} \frac{t^k \|A\|^k}{k!} \\
&= e^{t\|A\|} - 1
\end{aligned}$$

and $e^{t\|A\|} - 1 \rightarrow 0_+$ as $t \rightarrow 0$. Hence, it is uniformly continuous.

Note: A C_0 -semigroup is said to be uniformly continuous if and only if $\lim_{t \rightarrow 0} \|T(t) - I\| = 0$.

Corollary 3.3

Let $T(t) \supseteq \omega - OCP_n$ (ω -order preserving partial contraction mapping), then for each $f \in X, t \rightarrow T(t)$, f is the continuous function on $\mathbb{N} \rightarrow X$ if

- (i) $\lim_{t \rightarrow 0^+} T(t)f = f$ for each $f \in X$; and
- (ii) $\lim_{h \rightarrow 0} T(t+h) = T(t)$.

Proof

Let $t, h \geq 0$ and $f \in X$, then we have

$$\begin{aligned} \|T(t+h)f - T(t)f\| &= \|T(t)T(h)f - T(t)f\| \\ &\leq \|T(t)\| \|T(h)f - f\| \\ &\leq Me^{\omega t} \|T(h)f - f\| \end{aligned}$$

and for $t \geq h \geq 0$

$$\begin{aligned} \|T(t-h)f - T(t)f\| &= \|T(t-h)f - T(t-h)T(h)f\| \\ &= \|T(t-h)f - T(t-h)T(h)f\| \\ &\leq \|T(t-h)\| \|f - T(h)f\| \\ &\leq Me^{\omega t} \|f - T(h)f\| \end{aligned}$$

Therefore, the limit exists and the function is continuous on $\mathbb{N} \rightarrow X$.

Proposition 3.4

Let $T(t)$ be a C_0 -semigroup generated by A , where $A \in \omega - OCP_n$. Then

- (i) for each $f \in D(A)$, we have $T(t)f \in D(A)$; and
- (ii) for each $f \in D(A)$ and $T(t)f \in D(A)$ we have

$$T(t)f = AT(t) = T(t)Af$$

Proof

- (i) Let $f \in D(A)$ and fix $t \geq 0$ then $s > 0$ by definition of infinitesimal generator. The semigroup property $T(t)T(s) = T(s)T(t) = T(s+t)$ leads to

$$\begin{aligned} AST(t)f &= \frac{T(s)T(t)f - T(t)f}{s} \\ &= \frac{T(t)T(s)f - T(t)f}{s} \\ &= T(t) \cdot \frac{T(s)f - f}{s} \\ \lim_{s \rightarrow 0^+} T(t) \cdot \frac{T(s)f - f}{s} &\rightarrow T(t)(Af) \end{aligned}$$

we have that $f \in D(A)$ and $T(t)$ is continuous in X

$$\lim_{s \rightarrow 0^+} AST(t)f = T(t)Af = AT(t)f$$

hence

$$T(t)f \in D(A)$$

(ii) From $f \in D(A)$ and $h > 0$, we have

$$\begin{aligned}
 & \lim_{h \rightarrow 0^+} \frac{T(t+h) - T(t)}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{T(t+h)f - T(t)f}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{T(t)T(h)f - T(t)f}{h} \\
 & \lim_{h \rightarrow 0^+} \left(\frac{T(h) - I}{h} \right) T(t)f \\
 &= AT(t)f = T(t)Af.
 \end{aligned}$$

Consider

$$\begin{aligned}
 & \lim_{h \rightarrow 0^+} \frac{T(t-h)f - T(t)f}{h} = \lim_{h \rightarrow 0^+} \frac{T(t-h)f - T(t+h-h)f}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{T(t-h)f - T(t-h)T(h)f}{h} = \lim_{h \rightarrow 0^+} \left(\frac{I - T(h)}{h} \right) T(t-h)f \\
 &= AT(t)f.
 \end{aligned}$$

CONCLUSION

In this paper, it has been shown that $\omega - OCP_n$, which is a semigroup of linear operator has a common unique fixed point when it is considered as a weakly compatible function.

Conflict of Interest The authors declare that there are no conflicts of interest regarding the publication of the paper.

REFERENCES

- [1] Altun I., Turkoglu D. and Rhoades B. E. (2007). Fixed points of weakly compatible maps satisfying a general contractive condition of integral type. Fixed Point Theory Appl. Art. ID 17301.
- [2] Abbas M. and Jumiye G. (2008). Common fixed point result for non-commuting mappings without continuity in cone metric spaces. J. Math. Anal. Appl. **341**, 416-420.
- [3] Alata S. M., Rauf K. and Wahab O. T. (2015). Some results on common fixed point for generalized f-contraction mapping. Global Journal of Mathematics. **2** (1), 99-108.
- [4] Banach S. (1992). Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fund. Math. **3**, 133-181.
- [5] Vrabie I. I. (2003). C_0 -semigroup and application. Mathematics Studies, Elsevier, North-Holland **191**.
- [6] Rauf K. and Akinyele A. Y. (2019). Properties of ω -order-preserving partial contraction mapping and its relation to C_0 -semigroup, International Journal of Mathematics and Computer Science. **14** (1), 61-68.
- [7] Rauf K. and Yusuf O. Olaniyi (2019). Fixed point results in vector metric space, Octagon Mathematical Mag. (OMM). **27** (3). In Press.