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Enhancing Volatility Forecasting: A Comparative Study of GARCH(1,1) Models with Non-Normal Error Distributions

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Abstract

This study proposes a GARCH$(1,1)$ model with generalized logistic distribution (GLD) errors to better capture skewness and kurtosis in financial returns. Using crude oil price data, it compares GLD with GED and Student's t-distributions. Results show the GLD-based model outperforms others in forecasting accuracy and volatility modeling.

1. Introduction and Preliminaries

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model, introduced by [11], has proven to be an effective tool for modeling time-varying volatility in financial time series, especially in the presence of high kurtosis and persistent autocorrelation in squared returns. Accurate volatility modeling plays a fundamental role in areas such as asset pricing, risk management, and portfolio allocation [16] and [14] originally introduced the Autoregressive Conditional Heteroskedasticity (ARCH) framework, which allows the conditional variance to evolve over time as a function of past forecast errors, reflecting real-world phenomena like volatility clustering observed in financial markets. Bollerslev later extended this concept through the GARCH model, enabling more efficient modeling of the dynamic behavior of conditional variance. A wide range of studies

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[22], [20], [28] and [27] have demonstrated that the GARCH$(1,1)$ specification is particularly effective for capturing and forecasting volatility patterns in financial data. Despite its popularity and practical success, the GARCH model has limitations notably its reliance on distributional assumptions for the error term. Financial return series typically exhibit non-normal characteristics such as excess kurtosis and skewness, which the traditional normal distribution fails to capture. Mis-specification of the error distribution can significantly degrade forecast accuracy and lead to incorrect inferences about financial risk, as highlighted by [7]. To address this, alternative distributions such as the Student's t, skewed Student's t, and the Generalized Error Distribution (GED) have been employed, but these models often still fall short in fully capturing the heavy-tailed and asymmetric nature of real-world financial data. The pioneering work of [18], [15] and [12] brought attention to key features of financial time series, including volatility clustering, leptokurtosis, and asymmetry. Since then, several extensions of the GARCH framework have been proposed including the EGARCH [20], GJR-GARCH [17], APGARCH [29], TGARCH [31] and DAGARCH [24] models each aiming to address these stylized facts. Building on this line of research, this study proposes the use of the Type 1 Generalized Logistic Distribution (GLD) as an alternative error distribution within the GARCH$(1,1)$ framework. The GLD offers greater flexibility due to its ability to model both skewness and heavy tails, making it well-suited for financial return data. Through empirical analysis of crude oil price returns, we evaluate the predictive performance of GARCH$(1,1)$ models under various distributional assumptions, demonstrating that the GLD-enhanced GARCH model provides superior forecasting accuracy compared to traditional alternatives. Further information could be sourced from [1, 2, 3, 4, 5, 6], [8, 9, 10], [13], [19], [21], [23], [25], [26] and [32].

2. Methodology

The GARCH model was introduced by [11] and [30]. The model allows the conditional variance to depend on the previous lags.

2.1. Specification of GARCH Models.

$$r_t = \mu + \epsilon_t \tag{1}$$

$$\epsilon_t = z_t \sigma_t \tag{2}$$

$$z_t \sim N(0,1) \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{3}$$

For the purpose of this paper, we consider GARCH (1,1) Model specified as:

$$\sigma_t^2 = \omega + \alpha_1\epsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2$$

where σ_t^2 is the conditional variance, ω is the constant term, α_1 is ARCH parameter, β_1 is GARCH parameter and $\alpha_1 + \beta_1$ is volatility persistence parameter. The stability of GARCH$(1,1)$ model is guaranteed if $\alpha_1 + \beta_1 < 1$. Accordingly, volatility in financial markets will persist if the ARCH parameter represented by α_1 and GARCH parameter represented by β_1 are significant and their sum is close to 1 due to volatility clustering and leverage effect.

2.2. **Specification of Error Distribution for GARCH models.** In this paper, the forecast performance of GARCH model is evaluated under three distributional assumptions namely symmetric student-t distribution, generalized error distribution (GED) and Type 1 generalized logistic distribution (GLD).

2.2.1. *Symmetric Student-t Distribution.* Bollerslev proposed the use of Student-t distribution with $\nu > 0$ degrees of freedom. This distribution captures the heavy-tails in financial time series and the probability density function is given by:

$$f_y(y,\nu) = \frac{\Gamma\frac{\nu+1}{2}}{\sqrt{\pi(\nu-2)}\Gamma\frac{\nu}{2}(1+\frac{y^2}{\nu-2})^{\frac{\nu+1}{2}}} \quad (4) \tag{4}$$

where Γ is the usual Gamma function and $\nu > 2$ is the number of degrees of freedom.

2.2.2. *Generalized Error Distribution (GED).* This distribution comes from a family of exponential distributions which accounts for skewness and kurtosis that characterizes financial time series. The probability density function for the error term ϵ_t in the GARCH process under GED is defined as

$$f_y(y,\nu,d) = \frac{\nu}{d \times 2^{(\frac{\nu+1}{\nu})}\Gamma\frac{1}{\nu}} e^{-\frac{1}{2}|\frac{y}{d}|^{\nu}} \quad (5) \tag{5}$$

where

$$d = \left[\frac{2^{-\frac{2}{\nu}}\Gamma(\frac{1}{\nu})}{\Gamma_{\frac{3}{2}}}\right]^{\frac{1}{2}},$$

where y is the random variable, ν is the shape parameter, and d is the scale parameter.

2.2.3. *Type 1 Generalized Logistic Distribution.* Type 1 generalized logistic distribution is also called proportional reversed hazard logistic distribution, its density function may be asymmetric but it is always unimodal and log-concave. It has location, scale and skewness parameter, it is positively skewed for $\alpha > 1$ and negatively skewed for $0 < \alpha < 1$ and for $\alpha = 1$ the type 1 generalized logistic distribution coincides with the standard logistic distribution and it is symmetric. Its density function is log-concave for all values of α and the skewness parameter make it more flexible to model the data exhibiting unimodal density having some skewness present. The probability density function of the random variable under Type 1 Generalized Logistic Distribution is given as:

$$f(y,\alpha) = \frac{\alpha e^{-y}}{(1+e^{-y})^{\alpha+1}} \quad (6) \tag{6}$$

where y is the random variable, and α is the skewness parameter which control the tail region of the distribution.

3. Data Analysis

The time series data used for modeling volatility in this paper is monthy crude oil price from January, 2006 to June 2019 resulting in a total of 161 observations.

3.1. **Descriptive Statistics for Crude Oil Price.** The findings in Table 1 show that absolute returns for crude oil price was negatively skewed with a skewness value of -0.3126. By implication from the descriptive statistics, investors can expect frequent small gains and losses. The kurtosis for absolute returns was 6.7863. This indicates that the kurtosis of absolute returns is more leptokurtotic than the kurtosis of returns. The null hypothesis of normality for the Jarque-Bera test was rejected for crude oil price.

Table 1. Descriptive Statistics for Crude Oil Data

Statistic	**Value**
Mean	-0.0017
Median	0.0015
Minimum	-0.0164
Maximum	0.0145
Standard Deviation	0.0751
Skewness	-0.3126
Kurtosis	6.7863
Jarque-Bera	88.5420
Probability	0.0000
Sum of Square Deviation	0.0015
Observations	161

TABLE 2. ARCH LM Test: Heteroscedasticity Test

Statistics	**Result**	**Prob.**
F-Statistics	22.0769	0.000003
Obs* R-squared	20.8758	0.000005

From Table 2 above, it is evident that probability P of χ^2 test is less than significance level $\alpha = 0.05$ indicating that ARCH effects is present in crude oil price return series. Therefore, the null hypothesis of homoskedasticity of the residuals is rejected and the presence of time varying volatility clustering is accepted.

TABLE 3. Parameter Estimation of GARCH (1,1) Model under Different Error Distributions

Variables	μ	ω	α	β	$\alpha+\beta$	**Log-likelihood**
GED	96.440480	0.315200	0.135230	0.716468	0.8516	-3719.2350
Student t	75.366176	1.076861	0.925528	0.073471	0.9989	-9489.0740
GLD	75.359396	1.042324	0.923912	0.075079	0.9989	-1722.0650

From table 3 above, it is evident that the condition for stationarity of GARCH$(1, 1)$ model is satisfied since the sum of the ARCH parameter represented by α and GARCH parameter represented by β is less than 1 (Bera and Higgins, 1993). The measure of volatility persistence of GARCH (1, 1) model under three different error distribution specifications is presented in table 3 above. The volatility persistence of GARCH (1, 1) model denoted by $\alpha + \beta$ under GED, student t distribution and generalized logistic distribution is high with value 0.85, 0.99 and 0.99 respectively. These values indicate high volatility persistence. This findings further revealed a decaying volatility rate of 0.15, 0.01 and 0.01 for GARCH (1, 1) model with generalized error distributed innovation, GARCH (1, 1) model student t distributed innovation and GARCH (1, 1) model with generalized logistic distributed innovation respectively. Leverage or asymmetric effect means that negative returns tend to be associated with higher volatility than positive returns of the same magnitude. GARCH (1, 1) model with generalized logistic distribution has the highest likelihood. Therefore, based on this result GARCH (1, 1) model with generalized logistic distribution has the best forecast performance. It is evident from table 3 above that volatility will persist in crude oil market. By implications, higher expected volatility implies increased uncertainty and potential for larger price movements which can impact portfolio performance. To evaluate the predictive power of GARCH (1, 1) model under different distributional assumptions for the innovation, we employed four different loss functions to evaluate the predictive power of GARCH (1, 1) model. From Table 4 above, GARCH (1, 1) model with generalized logistic distributed innovation has the minimum value of the various loss functions considered. Hence, GARCH (1, 1) model with generalized logistic distributed innovation is the best forecasting model for returns series of crude oil price.

Table 4. Evaluation of Forecast Performance of GARCH(1,1) Model under Different Error Distribution

GARCH(1,1)	**Normal**	**GED**	**Student t-Distribution**	**GLD**
RMSE	0.5370	0.2850	0.0740	0.0520
MAE	0.5330	0.2751	0.0785	0.0551
MAPE	0.5550	0.2653	0.0766	0.0575
Theil	0.5169	0.2857	0.0787	0.0583

Conclusion. This study evaluated the forecasting performance of the GARCH$(1,1)$ model under three different error distribution assumptions: the Generalized Error Distribution (GED), the Student's t-distribution, and the Generalized Logistic Distribution (GLD). The results clearly indicate that the GARCH$(1,1)$ model with GLD for the error term consistently outperforms its counterparts across multiple forecast accuracy metrics. The flexibility of the GLD in capturing both skewness and heavy tails makes it particularly well-suited for modeling the volatility of financial return series such as crude oil prices. Overall, the findings suggest that incorporating the GLD into the GARCH$(1,1)$ framework enhances predictive accuracy and offers a more robust approach for modeling volatility dynamics in financial markets.

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