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Derivation of a New One-Step Numerical Integrator for Solving First Order Ordinary Differential Equations

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Abstract

This article presents a new class of one-step second derivative hybrid block method to solve general first order initial value problems of ordinary differential equations. The method is derived by interpolating an approximated power series at one point and collocating its first and second derivatives at some carefully selected intra-step points. The method is implemented in a block form to overcome the setbacks of applying starting values and predictors. The basic properties of the method were analysed and found to be a method of order of accuracy 10 and A-stable. The efficacy of the method is tested on several first order ordinary differential equations which include autonomous and non-autonomous equations, stiff systems and oscillatory problems and results compared with exact solutions.

1. Introduction

This article presents a new one-step second derivative hybrid block method for solving general first-order IVPs of the form:

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$$x' = f(t,x),\ x(t_0) = x_0,\ \forall t_0 \le t \le t_N \tag{1}$$

with f assumed to be continuous on $[t_0, t_N]$and existence of a unique solution is satisfied. Many fields of application, notably spring and damping system, control system, chemical reaction, electrical circuits, diffusion and control theory, yield initial value problems of the form (1). However, many of these problems do not often have theoretical solution, hence seeking numerical approximation is of utmost importance. As a result, most famous numerical methods such as Euler forward and Euler backward methods, Adams Bashforth and Adams Moulton methods, linear multistep methods, Runge-Kutta, Heun methods and among other methods have been proposed by various experts in the field of Mathematics. See – [5, 6, 8, 11, 14, 15, 16, 17]. Also over the years, multi-derivatives of solution and application of off-step points have been considered by various notable researchers in order to develop effective numerical methods for (1), see [1, 3, 7, 9, 18].
The idea of this article is to derive a new continuous form of one-step second derivatives hybrid block method using the power series as basis function and implement the new method in block form to produce approximate solution to (1).

2. Derivation of the Method

The method is derived by assuming an approximate solution of the form:

$$X(t) = \sum_{j=0}^{r+2s-1} a_j t^j \tag{2}$$

where a_j's are unknown coefficients of the power series function in (2), r and s are numbers of interpolation and collocation points respectively. Interpolating at t_n and collocating the first and second derivatives of (2) at $t_n, t_{n+\frac{5}{34}}, t_{n+\frac{1}{2}}, t_{n+\frac{29}{34}}, t_{n+1}$. These lead to

$$\begin{cases} X(t_n) = x_n \\ X'(t_{n+j}) = f_{n+j} \\ X''(t_{n+j}) = g_{n+j} \\ j = 0, \frac{5}{34}, \frac{1}{2}, \frac{29}{34}, 1; \end{cases} \tag{3}$$

and the coefficient matrix is given as:

(4)

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 \\ 0 & 1 & 2A & 3A^2 & 4A^3 & 5A^4 & 6A^5 & 7A^6 & 8A^7 & 9A^8 & 10A^9 \\ 0 & 1 & 2B & 3B^2 & 4B^3 & 5B^4 & 6B^5 & 7B^6 & 8B^7 & 9B^8 & 10B^9 \\ 0 & 1 & 2C & 3C^2 & 4C^3 & 5C^4 & 6C^5 & 7C^6 & 8C^7 & 9C^8 & 10C^9 \\ 0 & 1 & 2D & 3D^2 & 4D^3 & 5D^4 & 6D^5 & 7D^6 & 8D^7 & 9D^8 & 10D^9 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 \\ 0 & 0 & 2 & 6A & 12A^2 & 20A^3 & 30A^4 & 42A^5 & 56A^6 & 72A^7 & 90A^8 \\ 0 & 0 & 2 & 6B & 12B^2 & 20B^3 & 30B^4 & 42B^5 & 56B^6 & 72B^7 & 90B^8 \\ 0 & 0 & 2 & 6C & 12C^2 & 20C^3 & 30C^4 & 42C^5 & 56C^6 & 72C^7 & 90C^8 \\ 0 & 0 & 2 & 6D & 12D^2 & 20D^3 & 30D^4 & 42D^5 & 56D^6 & 72D^7 & 90D^8 \end{pmatrix}$$

where, $A = (x_{n+\frac{5}{34}}); B = (x_{n+\frac{1}{2}}); C = (x_{n+\frac{29}{34}})$ and $D = (x_{n+1})$ which is solved using matrix inversion method to obtain a_j's and then substituted into (2) to get the continuous scheme of the form:

(5)

$$\begin{aligned} X(t) = x_n + h\left(\beta_0(t) f_n + \beta_{\frac{5}{34}}(t) f_{n+\frac{5}{34}} + \beta_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + \beta_{\frac{29}{34}}(t) f_{n+\frac{29}{34}} + \beta_1(t) f_{n+1}\right) \\ + h^2\left(\gamma_0(t) g_n + \gamma_{\frac{5}{34}}(t) g_{n+\frac{5}{34}} + \gamma_{\frac{1}{2}}(t) g_{n+\frac{1}{2}} + \gamma_{\frac{29}{34}}(t) g_{n+\frac{29}{34}} + \gamma_1(t) g_{n+1}\right) \end{aligned}$$

Evaluating (5) at $t = t_{n+\frac{5}{34}}, t_{n+\frac{1}{2}}, t_{n+\frac{29}{34}}$ and t_{n+1}, the block algorithms are given as follows:

(6)

$$\left.\begin{aligned} x_{n+\frac{5}{34}} = x_n + \tfrac{48236526812245}{791200932938304} h f_n + \tfrac{5815453137955}{69329555914752} h f_{n+\frac{5}{34}} + \tfrac{575288125}{463713937344} h f_{n+\frac{1}{2}} \\ - \tfrac{29906050595}{69329555914752} h f_{n+\frac{29}{34}} + \tfrac{1109311642315}{791200932938304} h f_{n+1} + \tfrac{1815222121275}{163696744745856} h^2 g_n \\ - \tfrac{23029130875}{6773577302016} h^2 g_{n+\frac{5}{34}} - \tfrac{12819468125}{24024798277632} h^2 g_{n+\frac{1}{2}} - \tfrac{4299025}{14757249024} h^2 g_{n+\frac{29}{34}} \\ - \tfrac{9971114725}{163696744745856} h^2 g_{n+1} \end{aligned}\right\}$$

(7)

$$\left.\begin{aligned} x_{n+\frac{1}{2}} = x_n - \tfrac{2229002917}{20486760000} h f_n + \tfrac{531827571502403}{2548880732160000} h f_{n+\frac{5}{34}} + \tfrac{269281}{1632960} h f_{n+\frac{1}{2}} - \\ \tfrac{2082565315751}{509776146432000} h f_{n+\frac{29}{34}} + \tfrac{89027711}{4097352000} h f_{n+1} + \tfrac{2540591}{847728000} h^2 g_n + \\ \tfrac{37985434321}{2929747968000} h^2 g_{n+\frac{5}{34}} - \tfrac{82801}{4976640} h^2 g_{n+\frac{1}{2}} - \tfrac{103649561}{21701836800} h^2 g_{n+\frac{29}{34}} - \\ \tfrac{157309}{169545600} h^2 g_{n+1} \end{aligned}\right\}$$

(8)
$$\left.\begin{aligned} x_{n+\frac{29}{34}} = x_n + \tfrac{2618145660737}{20275557960000}hf_n + \tfrac{364210994519}{1776660480000}hf_{n+\frac{5}{34}} + \tfrac{761804585009}{2318569686720}hf_{n+\frac{1}{2}} + \\ \tfrac{214416025769}{1776660480000}hf_{n+\frac{29}{34}} + \tfrac{1410449179487}{20275557960000}hf_{n+1} + \tfrac{94010208599}{24330669552000}h^2g_n + \\ \tfrac{1033435097}{59222016000}h^2g_{n+\frac{5}{34}} - \tfrac{12819468125}{24024798277632}h^2g_{n+\frac{1}{2}} - \tfrac{788317237}{37287936000}h^2g_{n+\frac{29}{34}} - \\ \tfrac{68512139849}{24330669552000}h^2g_{n+1} \end{aligned}\right\}$$

(9)
$$\left.\begin{aligned} x_{n+1} = x_n + \tfrac{83566921}{640211250}hf_n + \tfrac{1018388173679}{4978282680000}hf_{n+\frac{5}{34}} + \tfrac{269281}{816480}hf_{n+\frac{1}{2}} + \\ \tfrac{1018388173679}{4978282680000}hf_{n+\frac{29}{34}} + \tfrac{83566921}{640211250}hf_{n+1} + \tfrac{103973}{26491500}h^2g_n \\ + \tfrac{203039551}{11444328000}h^2g_{n+\frac{5}{34}} - \tfrac{203039551}{11444328000}h^2g_{n+\frac{29}{34}} - \tfrac{103973}{26491500}h^2g_{n+1} \end{aligned}\right\}$$

Following [10] and [13], let the linear operator defined on the methods (6) to (9) be

$$L\left[y\left(x\right);h\right] = \sum_{j=0}^{k}\left(\alpha_j y_{n+j} - h\left\{\beta_j f_{n+j}\right\} - h^2\left\{\gamma_j g_{n+j}\right\}\right) \tag{10}$$

Expanding the terms in (10) we get

$$L\left[y\left(t\right);h\right] = E_0 y(x) + E_1 hy(t) + \cdots + E_q h^q y(t) + \cdots \tag{11}$$

where the constant $E_q, \quad q = 0, 1, 2\cdots$ are given below:

$$\left.\begin{aligned} E_0 &= \textstyle\sum_{j=0}^{k}\alpha_j \\ E_1 &= \textstyle\sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k}\beta_j \\ E_2 &= \tfrac{1}{2!}\textstyle\sum_{j=1}^{k}\left(j\right)^2\alpha_j - \left(\sum_{j=1}^{k} j\beta_j + \sum_{j=0}^{k}\gamma_j\right) \\ &\vdots \\ E_q &= \tfrac{1}{q!}\textstyle\sum_{j=1}^{k}\left(j\right)^q\alpha_j - \tfrac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\beta_j - \tfrac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-2}\gamma_j \end{aligned}\right\} \tag{12}$$

$q = 3, 4, \cdots$

According to [12], we say the method is of order p if $E_0 = E_1 = ...E_q = 0$, $E_{q+1} \neq 0$. E_{q+1} is the error constant.

From our analysis, the methods (6) to (9) have the following order and error constants summarized in table 1.

Table 1. Order and Error Constants for the method

Equation	Order	Error constants, E_{q+1}
(6)	10	$\frac{54355055225}{882539920625142485680128}$
(7)	10	$\frac{186341}{26884497449792000}$
(8)	10	$\frac{146131455427861}{1103174900781428107100016000}$
(9)	10	$\frac{186341}{134422487248896000}$

In what follows, the methods (6) to (9) can generally be written as a matrix difference equation as follows:

$$P^{(1)}Y_w = P^{(0)}Y_{w-1} + h\left[Q^{(0)}F_{w-1} + Q^{(1)}F_w\right] + h^2\left[R^{(0)}G_{w-1} + R^{(1)}G_w\right] \tag{13}$$

where

$$Y_w = \left(y_{n+\frac{5}{34}}, y_{n+\frac{1}{2}}, y_{n+\frac{29}{34}}, y_{n+1}\right)^T; \quad Y_{w-1} = \left(y_{n-\frac{29}{34}}, y_{n-\frac{1}{2}}, y_{n-\frac{29}{34}}, y_n\right)^T$$

$$F_w = \left(f_{n+\frac{5}{34}}, f_{n+\frac{1}{2}}, f_{n+\frac{29}{34}}, f_{n+1}\right)^T; \quad F_{w-1} = \left(f_{n-\frac{69}{74}}, f_{n-\frac{3}{4}}, f_{n-\frac{1}{2}}, f_{n-\frac{1}{4}}, f_{n-\frac{5}{74}}, f_n\right)^T$$

$$G_w = \left(g_{n+\frac{5}{34}}, g_{n+\frac{1}{2}}, g_{n+\frac{29}{34}}, g_{n+1}\right)^T; \quad G_{w-1} = \left(g_{n-\frac{29}{34}}, g_{n-\frac{1}{2}}, g_{n-\frac{5}{34}}, g_n\right)^T$$

and $P^{(1)}, P^{(0)}, Q^{(1)}, Q^{(0)}, R^{(1)}, R^{(0)}$ are matrices whose elements are the coefficients of the block method.

3.2. Zero stability: Let

$$P^{(1)}Y_w - P^{(0)}Y_{w-1} = 0 \tag{14}$$

given by the characteristic polynomial $\rho(\lambda)$

$$\rho(\lambda) = |\lambda P^{(1)} - P^{(0)}| = \lambda^3(\lambda - 1) \tag{15}$$

Following [10] the block method (14) is zero-stable, since from (15) $\rho(\lambda) = 0$ satisfies $|\lambda_j| \leq 1, \quad j = 1, 2, 3, \cdots$ and for the roots whose $|\lambda_j| = 1$, the multiplicity does not exceed one. With reference to [12], we can say that the block method is convergent.

3.3. Region of Absolute Stability**:** Following [4], we apply the usual test equations

$$y' = \lambda y, \qquad y'' = \lambda^2 y$$

to yield

$$Y_w = \sigma(z)\, Y_{w-1}, \quad z = \lambda h,$$

where the matrix $\sigma(z)$ is given by

$$\sigma(z) = \left(P^{(1)} - zQ^{(1)} - z^2R^{(1)}\right)^{-1}\left(P^{(0)} - Q^{(0)} - R^{(0)}\right) \tag{16}$$

The Matrix $\sigma(z)$has eigenvalues $\{0, 0, 0, \lambda_4\}$. And the dominant eigenvalue $\lambda_4 : \mathrm{C} \to \mathrm{C}$ is a rational function given by

$$A(z) = \frac{-B(z)}{C(z)} \tag{17}$$

where $B(z) =$

$$\begin{pmatrix} 336271908368578987773 75z^8+2342303721038510171638400z^7+ \\ 781044824408749858395241 20z^6+1737438191078010953421660672z^5 \\ +257919225326009788957130027 52z^4+269334086656443554609452154880z^3 \\ +188142656770576300519403228 1600z^2+80668038370724022593651834880 00z \\ +161336076741448045187303669760 00 \end{pmatrix}$$

$$C(z) = 1663499961502784160 \begin{pmatrix} 21025z^8-1384170z^7+47476092z^6-1038752880z^5+15555540960z^4 \\ -161689812480z^3+1131595637760z^2-4849296076800z \\ +9698592153600 \end{pmatrix}$$

It is clear from the stability function (16) that for $Re(z) < 0$, $|\lambda_4| \leq 1$. Thus the block method is A-stable since its region of absolute stability contains the left half-plane C.

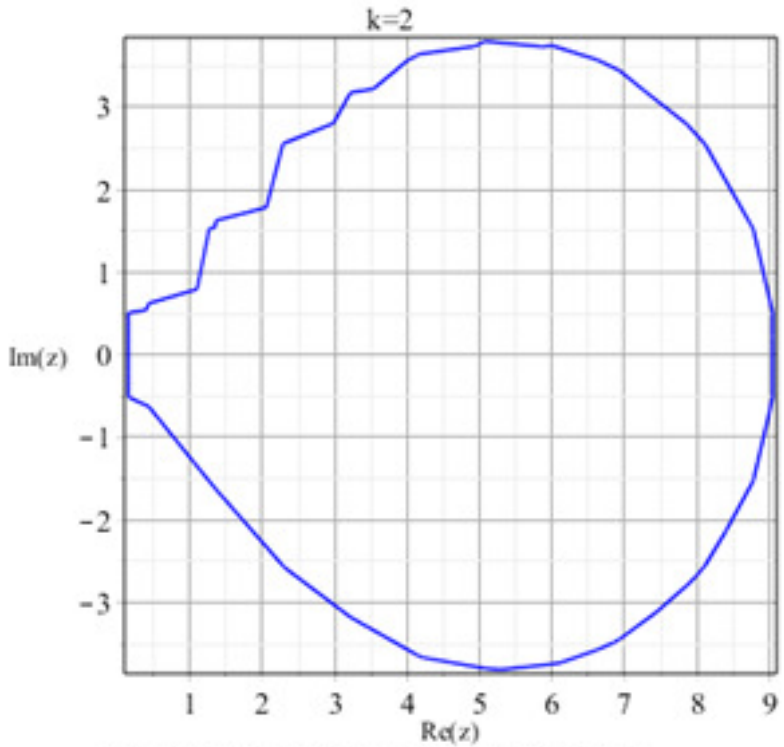


Fig. 1: Stability Region of the Block Method

2. Implementation

Our method is implemented as a block to solve some first order ODEs which requires no starting values and predictors. Initial conditions were obtained over sub-intervals $[x_0, x_1], \cdots, [x_{N-1}, x_N]$. For instance when $n = 0, (y_{\frac{5}{34}}, y_{\frac{1}{2}}, y_{\frac{29}{34}}, y_1)$ are obtained simultaneously over the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP, for $n = 1, (y_{\frac{39}{34}}, y_{\frac{3}{2}}, y_{\frac{63}{34}}, y_2)$ are also obtained simultaneously over the sub-interval $[x_1, x_2]$, as y_1 is now known from the previous block, and so on. Therefore, the sub-interval $[x_n, x_{n+1}]$ do not overlap.

2.1. **Numerical experiment.** Problem 1

$x' = -x(t); \quad x(0) = 1, \quad 0 \leq t \leq 1, \quad x(t) = e^{-t}$

Table 1. Numerical results for problem 1 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.1	0.90483741803595957316424911	0.90483741803595957316426161	$1.25*10^{-23}$
0.2	0.81873075307798185866993551	0.81873075307798185866995821	$2.27*10^{-23}$
0.3	0.74081822068171786606687381	0.74081822068171786606690461	$3.08*10^{-23}$
0.4	0.67032004603563930074443291	0.67032004603563930074447021	$3.73*10^{-23}$
0.5	0.60653065971263342360379951	0.60653065971263342360384171	$4.22*10^{-23}$
0.6	0.54881163609402643262845891	0.54881163609402643262850481	$4.59*10^{-23}$
0.7	0.49658530379140951470480011	0.49658530379140951470484851	$4.84*10^{-23}$
0.8	0.44932896411722159143010241	0.44932896411722159143015251	$5.01*10^{-23}$
0.9	0.40656965974059911188345421	0.40656965974059911188350531	$5.11*10^{-23}$
1.0	0.36787944117144232159552381	0.36787944117144232159557511	$5.13*10^{-23}$

Problem 2

$x'(t) = tx(t); \quad x(0) = 1, \quad 0 \leq t \leq 1, \quad x(t) = e^{\frac{t^2}{2}}$

TABLE 2. Numerical result for problem 2 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.1	1.00501252085940106338356 6	1.00501252085940106337629 1	$7.28*10^{-21}$
0.2	1.02020134002675581016014 4	1.02020134002675581012985 9	$3.03*10^{-20}$
0.3	1.04602785990871694271344 0	1.04602785990871694264065 5	$7.28*10^{-20}$
0.4	1.08328706767495855443598 8	1.08328706767495855429422 0	$1.42*10^{-19}$
0.5	1.13314845306682631682900 7	1.13314845306682631658029 9	$2.49*10^{-19}$
0.6	1.19721736312181016487682 4	1.19721736312181016446518 0	$4.12*10^{-19}$
0.7	1.27762131320488661066991 7	1.27762131320488661001140 6	$6.59*10^{-19}$
0.8	1.37712776433595708452687 7	1.37712776433595708349445 3	$1.03*10^{-18}$
0.9	1.49930250005676686965313 1	1.49930250005676686805314 7	$1.60*10^{-18}$
1.0	1.64872127070012814684865 1	1.64872127070012814438419 2	$2.46*10^{-19}$

Problem 3

$$y' = -10\,(y-1)^2\,; \qquad y\,(0) = 2, \quad 0 \leq x \leq 1,\ y\,(x) = 1 + \frac{1}{1+10x}$$

TABLE 3. Numerical result for problem 3 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.1	1.50000000000000000000000 0	1.50000037390908153796349 9	$3.746*10^{-7}$
0.2	1.33333333333333333333333 3	1.33333350026755019793372 0	$1.67*10^{-7}$
0.3	1.25000000000000000000000 0	1.25000009391404148572162 4	$9.39*10^{-8}$
0.4	1.20000000000000000000000 0	1.20000006010566341686636 1	$6.01*10^{-8}$
0.5	1.16666666666666666666667	1.16666670840677123894141 0	$4.17*10^{-8}$
0.6	1.14285714285714285714285 7	1.14285717352334961670731 5	$3.07*10^{-8}$
0.7	1.12500000000000000000000 0	1.12500002347881548178343 6	$2.35*10^{-8}$
0.8	1.11111111111111111111111 1	1.11111112966227392533730 6	$1.86*10^{-8}$
0.9	1.10000000000000000000000 0	1.10000001502644172203715 7	$1.50*10^{-8}$
1.0	1.09090909090909090909090 9	1.09090910332763764548281 8	$1.24*10^{-8}$

Problem 4

$x' = -10tx; \qquad x(0) = 1, \;\; 0 \leq t \leq 1, \; x(t) = e^{-5t^2}$

TABLE 4. Numerical result for problem 4 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t) \rvert$
0.1	0.95122942450071400909142 53	0.95122942450070747741177 08	$6.53*10^{-15}$
0.2	0.81873075307798185866993 55	0.81873075307796465744374 91	$1.72*10^{-14}$
0.3	0.63762815162177329314374 34	0.63762815162175539965597 47	$1.79*10^{-14}$
0.4	0.44932896411722159143010 24	0.44932896411721360553009 91	$7.99*10^{-15}$
0.5	0.28650479686019010032488 54	0.28650479686019144811932 25	$1.35*10^{-15}$
0.6	0.16529888822158653829680 47	0.16529888822158941037527 85	$2.87*10^{-15}$
0.7	0.08629358649937051097207 352	0.08629358649937012170943 457	$3.89*10^{-16}$
0.8	0.04076220397836621516607 926	0.04076220397836369630668 346	$2.52*10^{-15}$
0.9	0.01742237463949351138695 445	0.01742237463949164358479 555	$1.87*10^{-15}$
1.0	0.00673794699908546709663 6048	0.00673794699908512817240 0253	$3.39*10^{-16}$

Problem 5

$x_1^{'} = -21x_1 + 19x_2 - 20x_3; \quad x_1(0) = 1;$
$x_2^{'} = 19x_1 - 21x_2 + 20x_3; \quad x_2(0) = 0;$
$x_3^{'} = 40x_1 - 40x_2 - 40x_3; \quad x_3(0) = -1;$

$0 \leq t \leq 3$

$x_1(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-40t}\sin(40t) + \frac{1}{2}e^{-40t}\cos(40t);$
$x_2(t) = \frac{1}{2}e^{-2} - \frac{1}{2}e^{-40t}\sin(40t) - \frac{1}{2}e^{-40t}\cos(40t);$
$x_3(t) = e^{-40t}\sin(40t) - e^{-40t}\cos(40t).$

TABLE 5. Numerical result for problem 5a (h=0.2)

t	$x_1(t)$	$x_1(t_n)$	$\lvert x_1(t) - x_1(t_n)\rvert$
0.2	0.335301564464643629996 9883	0.335301628403181707180 9733	$5.27*10^{-5}$
0.4	0.224664411973797304262 9092	0.224664411952865015177 2362	$2.80*10^{-7}$
0.6	0.150597105947014311655 4539	0.150597105936117256351 0877	$9.77*10^{-11}$
0.8	0.100948258997336478293 3505	0.100948258987615106055 0727	$1.18*10^{-13}$
1.0	0.067667641618306346113 05751	0.067667641610160803658 47104	$9.12*10^{-17}$
1.2	0.045358976644706251686 58251	0.045358976638154107212 33318	$1.59*10^{-17}$
1.4	0.030405031312608982497 81078	0.030405031307484943081 32720	$1.24*10^{-17}$
1.6	0.020381101989183107583 03963	0.020381101985257683197 32042	$9.53*10^{-18}$
1.8	0.013661861223646280400 78153	0.013661861220686078414 00442	$7.19*10^{-18}$
2.0	0.009157819444367090146 859010	0.009157819442162331555 717504	$5.35*10^{-18}$
2.2	0.006138669951534220589 469695	0.006138669949908537321 146704	$3.95*10^{-18}$
2.4	0.004114873524510014420 681014	0.004114873523321220148 046277	$2.89*10^{-18}$
2.6	0.002758282210380386208 996878	0.002758282209517107524 817900	$2.10*10^{-18}$
2.8	0.001848931858241465410 346318	0.001848931857618279094 777967	$1.51*10^{-18}$
3.0	0.001239376088333179211 522584	0.001239376087885606768 948808	$1.09*10^{-18}$

TABLE 6. Numerical result for problem 5b (h=0.2)

t	$x_2(t)$	$x_2(t_n)$	$\lvert x_2(t) - x_2(t_n)\rvert$
0.2	0.33535037428834497180866 51	0.33501841761631953919964 91	$3.85*10^{-4}$
0.4	0.22466451974417424400988 59	0.22466455214272125356497 63	$3.13*10^{-7}$
0.6	0.15059710593100098341989 48	0.15059710595433095454393 83	$6.35*10^{-11}$
0.8	0.10094825899732591355060 81	0.10094825898759754816719 02	$1.25*10^{-13}$
1.0	0.06766764161830634894644 666	0.06766764161016080332000 290	$1.28*10^{-16}$
1.2	0.04535897664470625168749 482	0.04535897663815410721434 232	$1.59*10^{-17}$
1.4	0.03040503131260898249781 037	0.03040503130748494308132 687	$1.24*10^{-17}$
1.6	0.02038110198918310758303 963	0.02038110198525768319732 027	$9.53*10^{-18}$
1.8	0.01366186122364628040078 153	0.01366186122068607841400 428	$7.19*10^{-18}$
2.0	0.00915781944436709014685 9010	0.00915781944216233155571 7491	$5.35*10^{-18}$
2.2	0.00613866995153422058946 9695	0.00613866994990853732114 6664	$3.95*10^{-18}$
2.4	0.00411487352451001442068 1014	0.00411487352332122014804 6246	$2.89*10^{-18}$
2.6	0.00275828221038038620899 6878	0.00275828220951710752481 7879	$2.10*10^{-18}$
2.8	0.00184893185824146541034 6318	0.00184893185761827909477 7953	$1.51*10^{-18}$
3.0	0.00123937608833317921152 2584	0.00123937608788560676894 8792	$1.09*10^{-18}$

Problem 6.

$x_1^{'} = -1002x_1 + 100x_2^2; \quad x_1(0) = 1;$
$x_2^{'} = x_1 - x_2(1 + x_2); \quad x_2(0) = 1;$
$x_1(t) = e^{-2t}; \quad x_2(t) = e^{-t};$
$0 \leq t \leq 1$

TABLE 7. Numerical result for problem 6a (h=0.02)

t	$x_1(t)$	$x_1(t_n)$	$\lvert x_1(t) - x_1(t_n)\rvert$
0.1	0.818730753077981858669935508619 03	0.818730753077981858669935508642 16	2.31*10^{-29}
0.2	0.670320046035639300744432925147 82	0.670320046035639300744432925172 01	2.42*10^{-29}
0.3	0.548811636094026432628458917232 56	0.548811636094026432628458917256 40	2.38*10^{-29}
0.4	0.449328964117221591430102385015 56	0.449328964117221591430102385038 19	2.26*10^{-29}
0.5	0.367879441171442321595523770161 46	0.367879441171442321595523770182 39	2.09*10^{-29}
0.6	0.301194211912202096644977607083 22	0.301194211912202096644977607102 22	1.90*10^{-29}
0.7	0.246596963941606476939861239833 76	0.246596963941606476939861239850 77	1.70*10^{-29}
0.8	0.201896517994655408485179267643 34	0.201896517994655408485179267658 40	1.51*10^{-29}
0.9	0.165298888221586538296804720432 21	0.165298888221586538296804720445 41	1.32*10^{-29}
1.0	0.135335283236612691893999494972 48	0.135335283236612691893999494983 98	1.15*10^{-29}

TABLE 8. Numerical result for problem 6b (h=0.02)

t	$x_2(t)$	$x_2(t_n)$	$\lvert x_2(t) - x_2(t_n)\rvert$
0.1	0.904837418035959573164249059446 43	0.904837418035959573164249059450 19	3.76*10^{-30}
0.2	0.818730753077981858669935508619 03	0.818730753077981858669935508625 64	6.61*10^{-30}
0.3	0.740818220681717866066873779317 81	0.740818220681717866066873779326 52	8.71*10^{-30}
0.4	0.670320046035639300744432925147 82	0.670320046035639300744432925158 02	1.02*10^{-29}
0.5	0.606530659712633423603799534991 18	0.606530659712633423603799535002 39	1.12*10^{-29}
0.6	0.548811636094026432628458917232 56	0.548811636094026432628458917244 40	1.18*10^{-29}
0.7	0.496585303791409514704800093397 52	0.496585303791409514704800093409 70	1.22*10^{-29}
0.8	0.449328964117221591430102385015 56	0.449328964117221591430102385027 83	1.23*10^{-29}
0.9	0.406569659740599111883454239645 62	0.406569659740599111883454239657 80	1.22*10^{-29}
1.0	0.367879441171442321595523770161 46	0.367879441171442321595523770173 41	1.20*10^{-29}

Problem 7

$x^{'} = -2\pi\sin(2\pi t) - \frac{1}{10^{-3}}(x - \cos(2\pi t)); \qquad x(t) = \cos(2\pi t);$

$x(0) = 1, 0 \leq t \leq 1$

TABLE 9. Numerical result for problem 7 (h=0.1)

t	$x(t)$	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.1	0.80901699437494742410229 35	0.80901699437494740446196 35	1.96*10^{-17}
0.2	0.30901699437494742410229 38	0.30901699437494736774798 71	5.64*10^{-17}
0.3	-0.30901699437494742410229 34	-0.30901699437494750228633 66	7.82*10^{-17}
0.4	-0.80901699437494742410229 30	-0.80901699437494749604711 13	7.19*10^{-17}
0.5	-1.00000000000000000000000	-1.00000000000000000387101 10	3.87*10^{-17}
0.6	-0.80901699437494742410229 35	-0.80901699437494741492281 14	9.18*10^{-18}
0.7	-0.30901699437494742410229 41	-0.30901699437494737057487 96	5.35*10^{-17}
0.8	0.30901699437494742410229 21	0.30901699437494750152240 91	7.42*10^{-17}
0.9	0.80901699437494742410229 34	0.80901699437494749584067 64	7.17*10^{-17}
1.0	1.00000000000000000000000	1.00000000000000000386543 16	3.87*10^{-17}

Problem 8

$x^{'} = 5e^{5t}(x-t)^2 + 1; \quad x(0) = -1;$

$x(t) = t - e^{-5t}; \quad 0 \leq t \leq 1$

TABLE 10. Numerical result for problem 8 (h=0.1)

t	$x(t)$	$x_n(t)$	$\|x(t) - x_n(t)\|$
0.1	-0.50653065971263342360379 95	-0.50653065971263375314019 69	$3.30*10^{-16}$
0.2	-0.16787944117144232159552 38	-0.16787944117144264269911 80	$3.21*10^{-16}$
0.3	0.07686983985157017106671 95	0.07686983985156993170964 293	$2.39*10^{-16}$
0.4	0.26466471676338730810600 05	0.26466471676338714652194 39	$1.62*10^{-16}$
0.5	0.41791500137610120483047 13	0.41791500137610110078911 73	$1.04*10^{-16}$
0.6	0.55021293163213605702065 76	0.55021293163213599169598 77	$6.53*10^{-17}$
0.7	0.66980261657768149926021 37	0.66980261657768145882195 96	$4.04*10^{-17}$
0.8	0.78168436111126581970628 20	0.78168436111126579487874 27	$2.48*10^{-17}$
0.9	0.88889100346175769350385 69	0.88889100346175767833464 59	$1.52*10^{-17}$
1.0	0.99326205300091453290336 40	0.99326205300091452366210 44	$9.24*10^{-17}$

Problem 9

$x_1^{'} = -x_1 + 95x_2; \quad x(0) = 1;$
$x_2 = -x_1 - 97x_2; \quad x_2(0) = 1;$
$x_1(t) = \frac{95}{47}e^{-2t} - \frac{48}{47}e^{-96t};$
$x_2(t) = \frac{48}{47}e^{-96t} - \frac{1}{47}e^{-2t};$

TABLE 11. Numerical result for problem 9a (h=0.0625)

t	$x_1(t)$	$x_1(t_n)$	$\lvert x_1(t) - x_1(t_n)\rvert$
0.0625	1.781238843426735703571460	1.781235081308101235617369	$3.76*10^{-6}$
0.1250	1.574165520629585164451548	1.574165501965006985049470	$1.87*10^{-8}$
0.1875	1.389201718172411300331660	1.389201718102962425280066	$6.94*10^{-11}$
0.2500	1.225966227040172567257595	1.225966227039942867885903	$2.30*10^{-13}$
0.3125	1.081911398070203858018172	1.081911398070203145780032	$7.12*10^{-16}$
0.3750	0.954783457668008213710741 3	0.954783457668008211591532 1	$2.12*10^{-18}$
0.4375	0.842593444031027621659692 0	0.842593444031027621654506 9	$5.19*10^{-21}$
0.5000	0.743586104495468522372475 6	0.743586104495468522373416 7	$9.41*10^{-22}$
0.5625	0.656212434022196262355772 4	0.656212434022196262356723 9	$9.52*10^{-22}$
0.6250	0.579105440462086372997108 8	0.579105440462086372998041 8	$9.33*10^{-22}$
0.6875	0.511058757477679050894549 2	0.511058757477679050895454 9	$9.06*10^{-22}$
0.7500	0.451007770512783696780035 1	0.451007770512783696780907 0	$8.72*10^{-22}$
0.8125	0.398012960591911563310260 58	0.398012960591911563310343 94	$8.34*10^{-22}$
0.8750	0.351245204846644404992939 2	0.351245204846644404993731 2	$7.92*10^{-22}$
0.9375	0.309972805324855404533587 8	0.309972805324855404534336 8	$7.49*10^{-22}$
1.0000	0.273550040584642675104892 6	0.273550040584642675105597 7	$7.05*10^{-22}$

TABLE 12. Numerical result for problem 9b (h=0.0625)

t	$x_2(t)$	$x_2(t_n)$	$\lvert x_2(t) - x_2(t_n)\rvert$
0.0625	-0.016245038257544897841674 98	-0.016241276139 1042988730028	$3.76*10^{-6}$
0.1250	-0.016563954486775427961207 52	-0.016563935822 19724855862798	$1.87*10^{-8}$
0.1875	-0.014623160590466903241256 11	-0.014623160521 01802818899741	$6.94*10^{-11}$
0.2500	-0.012904907614905720049996 44	-0.012904907614 67602067752201	$2.30*10^{-13}$
0.3125	-0.011388541032223374104927 22	-0.011388541032 22266186592367	$7.12*10^{-16}$
0.3750	-0.010050352185978799434648 12	-0.010050352185 97879731452526	$2.12*10^{-18}$
0.4375	-0.008869404674010816489097 666	-0.008869404674 01081648297194	$6.13*10^{-21}$
0.5000	-0.007827222152583879181427 998	-0.007827222152 58387918142069	$7.30*10^{-24}$
0.5625	-0.006907499305496802761636 139	-0.006907499305 49680276164610	$9.97*10^{-24}$
0.6250	-0.006095846741706172347337 978	-0.006095846741 70617234734780	$9.82*10^{-24}$
0.6875	-0.005379565868186095272574 202	-0.005379565868 18609527258373	$9.53*10^{-24}$
0.7500	-0.004747450215924038913474 053	-0.004747450215 92403891348323	$9.18*10^{-24}$
0.8125	-0.004189610110727532980027 430	-0.004189610110 72753298003620	$8.78*10^{-24}$
0.8750	-0.003697317945754151631504 623	-0.003697317945 75415163151296	$8.34*10^{-24}$
0.9375	-0.003262871634998477942458 819	-0.003262871634 99847794246670	$7.88*10^{-24}$
1.0000	-0.002879474111417291316893 606	-0.002879474111 41729131690102	$7.42*10^{-24}$

Problem 10
$x^{'} = -\frac{x^3}{2}; \quad x(0) = 1;$
$x(t) = \frac{1}{\sqrt{1+t}}; \quad ;$
$0 \leq t \leq 4;$

TABLE 13. Numerical result for problem 10 (h=0.1)

t	x(t)	$x_n(t)$	$\lvert x(t) - x_n(t)\rvert$
0.4	0.845154254728516577509618 1	0.845154254728516635118679 6	5.76*10^{-17}
0.8	0.745355992499929898803058 0	0.745355992499929940146059 9	4.13*10^{-17}
1.2	0.674199862463242086246490 9	0.674199862463242116975526 4	3.07*10^{-17}
1.6	0.620173672946042280951277 8	0.620173672946042304884952 0	2.39*10^{-17}
2.0	0.577350269189625764509148 9	0.577350269189625783822022 8	1.93*10^{-17}
2.4	0.542326144546640430000133 7	0.542326144546640446007715 3	1.60*10^{-17}
2.8	0.512989176042577047728337 4	0.512989176042577061276280 2	1.35*10^{-17}
3.2	0.487950036474266589677192 3	0.487950036474266601336598 1	1.17*10^{-17}
3.6	0.466252404120156882818006 9	0.466252404120156892990191 3	1.02*10^{-17}
4.0	0.447213595499957939281834 7	0.447213595499957948258111 2	8.98*10^{-18}

Conclusions: One-Step Second Derivative Block intra-point Method for integrating general first order initial value problems of ordinary differential equations is proposed in this paper. The method which is found to be a uniform order of 10 and A-stable is implemented in block form to simultaneously solve IVPs over a partitioned interval of integration. We have demonstrated the efficiency of the method on ten numerical examples (which include linear and non-linear IVPs) and it competes favourably with the exact solution.

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