



CONJUGACY OF A SUBGROUP OF SYMMETRIC GROUP

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ABSTRACT

This paper studied conjugacy on a subgroup that is not invariant generated by plain hunting and other related properties.

1. INTRODUCTION

A subgroup H of a group G is defined in [3] as a normal(or invariant) subgroup of G if $gHg^{-1} = H$ for all $g \in G$, that is, if it's left and right cosets coincide. It has been pointed out before [6] that permutations were first studied in the 1600's in the context of the ringing of bells in a certain order and not (as usually stated) in the 1770's by Lagranges.

Gary, in his article [2], further provided a brief explanation of how permutation arise in the ringing of bells. The subgroup obtained by plain hunt method of ringing of bells in [2] is used in this paper for $n \geq 3$. The permutations A and B generate a subgroup H_n of S_n of order $2n$ and is obtained by referring to n bells listed in a particular order as row. For example, rounds is the row 123456 where $n = 6$. The permutation is done in such a way that the last element obtained is an identity element. He further stated that rows are compiled, subject to the following rules:

- The first and last rows must be rounds;
- No row may be repeated(apart from round which appear twice);
- Each bell may only change place by one position when moving from one row to the subsequent row;
- No bell occupies the same position for more than two successive rows.

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In his article, Peter Wenham[8] explained plain hunting of bells from rounds for four bells as:

Rounds	1	2	3	4
Change two pairs	2	1	4	3
Change the middle pair	2	4	1	3
Change the outside pairs	4	2	3	1
Change the middle pair	4	3	2	1
Change the outside pairs	3	4	1	2
Change the middle pair	3	1	4	2
Change the outside pairs	1	3	2	4
Change the middle pair	1	2	3	4

It can be seen that all the numbers(bells) follow similar pattern in plain hunting path. The same subgroup is generated following Gary's method in [2] of plain hunting, but the elements do not occupy same positions.

Change Ringing is a team sport, a highly coordinated musical performance, an antique art, and a demanding exercise that involves a group of people ringing rhythmically a set of tuned bells through a series of changing sequences that are determined by mathematical principles and executed according to learned patterns [9]. Church bells have been rung in England for almost 1000 years to call parishioners to church. Originally churches had just one bell, probably rung by the priest who conducted the service. In time more bells were bought and rung. About 400 years ago the idea of changing the order of the bells was suggested, with a conductor calling the changes to tell the ringers which bells to swap. Later, methods were developed where the order of the bells changes according to a pattern memorised by the ringers. Call Changes are often the first thing a beginner learns after rounds; they are still rung in most church towers from time to time. Bells are normally tuned to a major scale; some sequences sound more pleasing than others.

Bells are heavy, often very heavy; a ringer can only make a small alteration to the swing of a bell when it is in motion. However, because the bell can be held stationary at the balance, it is possible for the bells to change their striking order but each bell can only move by one position in the order. The convention for Call Changes is that the instruction to change is called by one of the ringers, often known as the Caller, near the beginning of a handstroke; during that stroke and the next backstroke the other ringers prepare for the new order which happens when the next handstroke is rung [10].

In a group G , two elements x and y are conjugate when $x = hyh^{-1}$ for some $h \in G$.

Conjugacy is studied on the subgroup H_n in this work. The conjugacy class of an element x in a group G is the set of elements conjugate to it and it is written as:

$$C_x = \{h x h^{-1} : h \in G\}.$$

That is, the collection of all conjugate elements together in a group is called a conjugate class. John and Mark [4] established a general result about the distribution of the conjugacy classes of a finite group G between the cosets of a normal subgroup H such that G/H is cyclic. Xiaolei, L., Yanming, W. and Huaguan, W. [7] investigated influences of lengths of conjugacy classes of finite groups on the structure of finite groups. Certain conditions were imposed on the lengths of the conjugacy classes of G and the group structure of G was described under these conditions [1]. Patrick [5] also studied the number of conjugacy classes in a finite group. The action of any group on itself by conjugation and the corresponding conjugacy relation play an important role in group theory.

2. MAIN RESULTS

Plain hunt on n bells uses two permutations denoted by A and B in the cycle forms.

For odd $n \geq 3, i = 1$ here,

$$A = (i \ i+1)(i+2 \ i+3) \dots (i+n-3 \ i+n-2)$$

$$B = (i+1 \ i+2)(i+3 \ i+4)(i+5 \ i+6) \dots (i+n-2 \ i+n-1)$$

For even $n \geq 3$,

$$A = (i \ i+1)(i+2 \ i+3)(i+4 \ i+5) \dots (i+n-2 \ i+n-1)$$

$$B = (i+1 \ i+2)(i+3 \ i+4)(i+5 \ i+6) \dots (i+n-3 \ i+n-2)$$

Plain Hunt on n bells uses two permutations applied alternately to rounds until rounds comes back again by Gary [2].

2.1 Examples:

1. Let $n = 3$ (odd)

$A = (12), B = (23)$; the elements in the subgroup, H_3 can be generated as:

1 2 3

$$A = (12)$$

2 1 3

$$AB = (132)$$

3 1 2

$$ABA = (13)$$

3 2 1

$$ABAB = (123)$$

2 3 1

$$ABABA = (23)$$

1 3 2

$$ABABAB = \text{identity}$$

1 2 3

2. Let $n = 8(\text{even})$; $A = (12)(34)(56)(78)$; $B = (23)(45)(67)$. The elements of the subgroup H_8 are thus generated:

1 2 3 4 5 6 7 8

$$A = (12)(34)(56)(78)$$

2 1 4 3 6 5 8 7

$$AB = (13578642)$$

3 1 5 2 7 4 8 6

$$ABA = (14)(36)(58)$$

4 2 6 1 8 3 7 5

$$ABAB = (1584)(2376)$$

5 3 7 1 8 2 6 4

$$ABABA = (16)(24)(38)(57)$$

6 4 8 2 7 1 5 3

$$ABABAB = (17438256)$$

7 5 8 3 6 1 4 2

$$ABABABA = (18)(26)(37)$$

8 6 7 4 5 2 3 1

$$ABABABAB = (18)(27)(36)(45)$$

8 7 6 5 4 3 2 1

$$ABABABABA = (17)(28)(35)(46)$$

7 8 5 6 3 4 1 2

$$ABABABABAB = (16528347)$$

6 8 4 7 2 5 1 3

$$ABABABABABA = (15)(27)(48)$$

5 7 3 8 1 6 2 4

$$ABABABABABAB = (1485)(2673)$$

4 6 2 8 1 7 3 5

$$ABABABABABABA = (13)(25)(47)(68)$$

3 5 1 7 2 8 4 6

$$ABABABABABABAB = (12468753)$$

$$\begin{array}{cccccccc} 2 & 4 & 1 & 6 & 3 & 8 & 5 & 7 \\ & & & & & & & & \text{ABABABABABABABA} = (23)(45)(67) \\ 1 & 3 & 2 & 5 & 4 & 7 & 6 & 8 \\ & & & & & & & & \text{ABABABABABABABAB} = (\text{identity}) \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

The permutations A and B generate a subgroup H_n of S_n of order $2n$ which [2] called the hunting subgroup.

Theorem 2.1: Let n be the number of bells. H_n is not a normal subgroup.

Proof: Let $a, b, c \in H_n$, where $n \in N$. S_n is the collection of all permutations of n and is a group under permutation multiplication. Permutations are functions and H_n is a set of functions.

To show that the operation performed to generate H_n is closed, then for $a, b, c \in H_n$

$$(a.b) = c \in H_n.$$

Also, $(a.b).c = a.(b.c) \in H_n$, then the function composition is associative. The permutation $i \in H_n$ such that $i.a = a.i = a$, for all $a \in H_n$ implies existence of identity in H_n .

Let a^{-1} be another permutation in H_n for each a , such that $aa^{-1} = a^{-1}a = i(identity)$. This established the existence of inverse for each element $a \in H_n$.

Let $s \in S_n$ and $h \in H_n, s.h = a$ and $h.s = b$, then $s.h \neq h.s \dots (1)$

Multiply both sides of (1) by s^{-1} from the right,

$$shs^{-1} \neq hss^{-1}$$

$shs^{-1} \neq hi$, i is the identity element,

$$shs^{-1} \neq h.$$

Generally for the subgroup H_n , sH_ns^{-1} is not contained in H_n .

Theorem 2.2: Every element with only product of transpositions(cycles of length two) is the inverse of itself by composition of mappings.

Proof: Let the product of transpositions

$$(b_1 \ b_2)(b_3 \ b_5)(b_4 \ b_6)(b_7 \ b_8) \dots (b_t \ b_k)$$

be an element obtained from plain hunting, where

$$b_1, b_2, b_3, b_4, b_5, b_6, \dots, b_t, b_k \in H_n.$$

The transpositions are in order of occurrence and a bell does not appear twice in a row.

$$\text{If } T = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \dots & b_t \\ b_3 & b_4 & b_1 & b_2 & b_5 \dots & b_x \end{pmatrix},$$

$$\text{then } T^2 = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_x \dots & b_t \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_x \dots & b_t \end{pmatrix}.$$

Corollary 2.1: H_n has n -even and n - odd permutations.

Proof: The subgroup H_n obtained by plain hunting, the method of changing pairs as seen in section 1 above has $2n$ elements for $n \geq 3$. S_n has a equal number

of both even and odd permutations, since H_n is a subgroup of S_n , H_n also has equal number of even and odd permutations.

Lemma 2.1: The lengths of cycles of elements in each conjugate class are equal.

Proof: This claim is obvious from the set of elements in each conjugacy class.

Lemma 2.2: Two permutations in H_n are conjugate if and only if they have the same cycle type but elements with the same cycle type may not necessarily conjugate.

Proof: The lengths of cycles of elements in each conjugacy class are equal from Lemma 2.1 implies that elements in a conjugacy class have the same cycle type. Choosing any two elements x and y of the same cycle type from H_n , is not a criterion for conjugacy.

Lemma 2.3: All cycles of the same length in H_n do not belong to the same conjugacy class.

Proof: The proof of this follows the reasoning of Lemma 2.2.

Theorem 2.3: Let $y \in H_n$ be a disjoint permutation of a particular cycle type. If for all $h \in H_n$, then $hyh^{-1} = x$, also a disjoint permutation of the same cycle type as y .

Proof: Assuming that

$$y = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_4 & b_3 & b_2 & b_1 \end{pmatrix}$$

with the cycle type $y = (b_1b_4)(b_2b_3)$

and

$$x = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_3 & c_4 & c_1 & c_2 \end{pmatrix},$$

for each $h \in H_n$, then $hyh^{-1} = x$, a disjoint permutation of the same cycle type. This implies that both x and y belong to the same conjugacy class.

Lemma 2.4: Any two elements of a conjugacy class have the same order but the converse is not true.

Proof: Let e be the identity element in H_n . If γ is a conjugacy class such that $x, y \in \gamma$ and α is the order of x and y , it implies that $x^\alpha = y^\alpha = e$. If the two elements x and y have the same order in G and $C_x = h x h^{-1}$; $C_y = g y g^{-1}$ for some $h, g \in G$. $x^\alpha = y^\alpha = e$ does not imply that $C_x = C_y$.

Lemma 2.5: The order of the subgroup H_n is the product of the order of each element of cycle length n and the number of elements in its conjugate class.

Proof: Let α be the order of each element $x \in H_n$ and k is the number of elements of a conjugate class each of which is of cycle length n . So $x^\alpha = kn$.

Lemma 2.6: Let G be a finite symmetric group of order $n!$ and H_n its subgroup. By Lagrange's theorem, $H_n \leq G$, then the quotient of $\frac{|G|}{|H_n|}$ is $\frac{(n-1)!}{2}$, $n \geq 3$.

Proof: Given that the order of group G is $n!$ and the order of H_n is $2n$, then the result follows.

Theorem 2.4: Any conjugate class whose elements are cycles of order n is abelian.

Proof: Let C_x be a conjugate class of H_n . Let $x, y \in C_x \subset H_n$ such that

$$x = \begin{pmatrix} a & b & c & d & e & f & g \\ c & a & e & b & g & d & f \end{pmatrix} \text{ and } y = \begin{pmatrix} a & b & c & d & e & f & g \\ b & d & a & f & c & g & e \end{pmatrix}.$$

$$\Rightarrow x = y^{-1}$$

$$xy = y^{-1}y$$

$$xy = 1(\text{identity element}).$$

$$\text{Also, } x^{-1} = y$$

$$x^{-1}x = yx$$

$$1 = yx$$

$$\Rightarrow xy = yx = 1.$$

Theorem 2.5: The order of each element in a conjugate class is the length of the cycle.

Proof: Let e be the identity element of H_n , cycle $(t_1, t_2, \dots, t_m)$ and C_x , a conjugate class of H_n . Let $x, y \in C_x$ such that $x = (t_1 t_3 t_4 t_2)$ then $x^4 = e$.

Theorem 2.6: Conjugacy classes of H_n are obtained using the odd and even occurrence of elements as: For odd n ,

$2k + 1; k = 0, 1, 2, 3, \dots, n - 1$ with n elements;

$2k, 2n - 2k; k = 1, 2, 3, \dots, n - 2$ with k elements for each n ;

and $2n$ is also a conjugate class.

For even n ,

$2k + 1; k = 0, 2, 4, \dots, n - 2$ with $\frac{n}{2}$ elements;

$2k + 1; k = 1, 3, 5, \dots, n - 1$ with $\frac{n}{2}$ elements;

$2k, 2n - 2k; k = 1, 2, 3, \dots, \frac{n-2}{2}$ with k elements for each n ;

n is a conjugate class and $2n$ is also a conjugate class.

Proof:

Let C_x denote a conjugate class of H_n . The status of an element of H_n is the position it occupies. The cardinality of H_n is $2n$. The conjugate classes for odd numbered and even numbered elements are obtained separately for each n .

For odd n ,

Let $H_n = \{a, b, c, d, e, \dots, 2n\}$ such that the position of a is 1, the position of b is 2, c is 3 ... Elements $a, c, e, \dots, 2n - 1$ are odd - numbered. If a is conjugate to c then

$c = hah^{-1}$, for some $h \in H_n$. A conjugate class is obtained containing odd positioned elements with the expression $2k + 1; k = 0, 1, 2, 3, \dots, n - 1$ and the class contains n elements. Elements with even positions form k conjugate classes with $\{2k, 2n - 2k\}$ elements in each class for $k = 1, 2, 3, \dots, n - 2$. The latter conjugate class has 2 elements in each class. The last conjugate class is the last element $2n$,

which is the identity element. Conjugate classes for odd n have three different characterization.

For even n ,

The elements of $H_{n(even)}$ can be subdivided into conjugate classes by splitting, using odd - even criterion. Elements having status(position) $2k + 1$, k is odd number up to $n - 1$, constitute a conjugate class with $\frac{n}{2}$ number of elements.

Another conjugate class also emanates by the elements occupying positions $2k, 2n - 2k, k = 1, 2, 3, \dots, \frac{n-2}{2}$. This implies that a conjugate class has only two elements with $\frac{n-2}{2}$ occurrences.

The last two conjugate classes for n - even are the identity element of status $2n$ and element of status n . Each of the two classes has an element each.

Theorem2.7: A conjugate class is a generator of other conjugate classes in S_n .

Proof: Let $x_1, x_2, x_3, \dots, x_n \in S_n$. If $x_1, x_2 \in S_n$ are conjugate elements defined as

$$x_1 = x_n x_2 x_n^{-1} \text{ for some } x_n \in S_n \dots (*)$$

Assuming that there are other elements in the same conjugate class with x_1 and x_2 . Then, from the right hand side of equation (*), for each $x_n \in S_n$, computing conjugate elements of x_2 implies:

$$x_1 x_2 x_n^{-1} = x_t$$

$$x_2 x_2 x_2^{-1} = x_r$$

$$\vdots$$

$$x_p x_2 x_p^{-1} = x_1$$

$$x_q x_2 x_q^{-1} = x_2.$$

This implies that x_t, x_r, x_1 and x_2 belong to the same conjugate class. The set x_t, x_r, x_1 and x_2 is a generator of another conjugate class which in turn is a generator of another conjugate class until all the conjugate classes are generated. Thus, other conjugate classes are obtained from one conjugate class.

3. SUMMARY

The hunting subgroup H_n is not an invariant and that Conjugacy classes of H_n are obtained using the odd and even occurrence of elements as: For odd n ,

$2k + 1; k = 0, 1, 2, 3, \dots, n - 1$ with n elements;

$2k, 2n - 2k; k = 1, 2, 3, \dots, n - 2$ with k elements for each n ;

and $2n$ is also a conjugate class.

For even n ,

$2k + 1; k = 0, 2, 4, \dots, n - 2$ with $\frac{n}{2}$ elements;

$2k + 1; k = 1, 3, 5, \dots, n - 1$ with $\frac{n}{2}$ elements;

$2k, 2n - 2k; k = 1, 2, 3, \dots, \frac{n-2}{2}$ with k elements for each n ;

n is a conjugate class and $2n$ is also a conjugate class.

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