DOI: https://doi.org/10.67897/njma.2022.ik5tsfsb

Nigerian Journal of Mathematics and Applications
Volume **32** (**II**), (2022), 120 – 128. *Printed by Unilorin press*
©*Nig. J. Math. Appl.* *http : //www.njmaman.com*

Special Issue in Honor of Prof. J. A. Gbadeyan's Retirement

A Zero-Stable Numerical Model as First, Second, and Third-Order Initial Value Problems Solver

Emmanuel Oluseye Adeyefa* and Oladapo Olakunle Kazeem

Abstract

This paper addresses solutions of multi-order ordinary differential equations (ODEs) using a single formulated scheme. The use of a block method to solve initial value problem of a specific order has attracted a lot of attention in the literature. The discrete method is formulated from continuous schemes obtained via collocation and interpolation techniques and applied in a block-by-block manner as a numerical integrator for first, second and third-order ODEs. The convergence properties of the method are discussed via zero-stability and consistency, and on investigation, the proposed method is found to be convergent. Solutions of the test problems considered are presented, as well as comparisons to existing approaches in the literature.

1. Introduction

This paper focuses on the development of

$$\begin{aligned} &\textstyle\sum_{i=0}^{k}\alpha_i(t)y_{n+i} = h\left(\sum_{i=0}^{k}\beta_i(t)f_{n+i} + \beta_j(t)f_{n+j}\right) \\ &+h^2\left(\sum_{i=0}^{k}\lambda_i(t)g_{n+i} + \lambda_j(t)g_{n+j}\right) + h^3\left(\sum_{i=0}^{k}\delta_i(t)w_{n+i} + \delta_j(t)w_{n+j}\right). \end{aligned} \tag{1.1}$$

to directly integrate ordinary differential equations (ODEs)

$$\begin{cases} y^{'}(x) = f(x, y(x)), y(x_0) = y_0 \\ y^{''}(x) = f(x, y(x), y^{'}(x)), y(x_0) = y_0, y^{'}(x_0) = y_0^{'} \\ y^{'''}(x) = f(x, y(x), y^{'}(x), y^{''}(x)), y(x_0) = y_0, y^{'}(x_0) = y_0^{'}, y^{''}(x_0) = y_0^{''} \end{cases} \tag{1.2}$$

Received: 11/06/2022, Accepted: 18/07/2022, Revised: 29/07/2022. * Corresponding author.
2015 *Mathematics Subject Classification.* 65L05 & 65L06.
Keywords and phrases. Block Method, Consistency, Convergence, Initial value problems, Zero-stability
Mathematics Department, Federal University Oye-Ekiti, Ekiti State, Nigeria
E-mails of the corresponding author: adeoluman@yahoo.com
ORCID of the corresponding author: 0000-0003-0942-6430

where in equation (1.1), either of $\alpha_0(t)$ and $\beta_0(t)$ do not varnish, $\alpha_k(t) = 1$, $\beta_k(t) \neq 0$ and $k = 1$. The traditional method of reducing higher-order ODEs to a set of first-order equations increases the problem's dimension and, as a result, requires more processing time. The use of block methods to solve equation (1.2) has been widely reported, and the resulting solutions have been thoroughly discussed by a number of scholars, including [5]. This block method approach, which generates approximations at different grid points within the integration interval without overlapping sub-intervals, has been reported to avoid the setback commonly experienced in reducing higher-order ODEs to a system of first-order equations and the predictor-corrector approach, see ([11], [6], [15], [2], [17] & [18]).

These efficient approaches have been directly applied as numerical integrators of ODEs focussing at solving IVPs of the same order, but using the same method to integrate IVPs of multiple order has not been widely reported. The goal of this study is to provide a self-starting approach for solving first, second, and third-order ODEs numerically.

In what immediately follows, the derivation of the proposed approach for direct integration of multi-order ODEs is discussed. The method's analysis and implementation are addressed in Section 3, and the results are discussed in Section 4. Finally, in Section 5, the paper's conclusion is presented.

2. Material and Method

In this section, the formulation of the proposed method is considered. We adopt the interpolation and collocation techniques employed in [4] and develop a novel numerical method for solving first, second, and third-order IVPs of ODEs. Using Chebyshev polynomials as the basis function, we develop a novel one-step three-hybrid approach for solving first, second, and third-order ODEs.

Thus,

$$y(x) = \sum_{j=0}^{k+8} a_j T_j \tag{2.1}$$

is introduced as an approximate solution to first, second and third-order ODEs of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \tag{2.2}$$

$$y''(x) = f(x, y(x), y'(x)), \quad y(x_0) = y_0, y'(x_0) = y'_0 \tag{2.3}$$

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \quad y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \tag{2.4}$$

Equation (2.1) is interpolated at$x = x_n$. Collocating the first and second derivatives of equation (2.1) at$x = x_{n+m}, m = 0, \frac{4}{5}, 1, \frac{6}{5}$and the third derivative at$x = x_{n+w}, w = 2$, we have

$$\begin{array}{l} \sum_{j=0}^{k+8} a_j T_j = y_n \\ \sum_{j=1}^{k+8} a_j T'_j = f_{n+v} \\ \sum_{j=2}^{k+8} a_j T''_j = g_{n+v} \\ \sum_{j=3}^{k+8} a_j T'''_j = X_{n+w} \end{array} \tag{2.5}$$

where; $T_j(x)$is the parameters of Chebyshev polynomials, g_{n+v} is the the first derivative of (2.1), $g_{n+v}$is its second derivative, $X_{n+w}$is its third derivative and k is the step number ($k = 2$).

Equation (2.5) is solved using the Gaussian elimination method to obtain the unknown variables $a's$ that are then substituted into equation (2.1). This results in an implicit continuous scheme of the form:

$$\begin{aligned}\alpha_0(t)y_n &= h\left(\sum_{j=0}^{1}\beta_j(t)f_{n+j}+\beta_{\frac{4}{5}}(t)f_{n+\frac{4}{5}}+\beta_{\frac{6}{5}}(t)f_{n+\frac{6}{5}}\right)\\ +h^2&\left(\sum_{j=0}^{1}\lambda_j(t)g_{n+j}+\lambda_{\frac{4}{5}}(t)g_{n+\frac{4}{5}}+\lambda_{\frac{6}{5}}(t)g_{n+\frac{6}{5}}\right)+h^3\left(\delta_2(t)X_{n+2}\right)\end{aligned} \tag{2.6}$$

where $t=\frac{2x-2x_n-h}{h}$, $\alpha_0(t)=1$. Then,

$$A=BC$$

where

$$A=\begin{pmatrix}\beta_0\\ \beta_{\frac{4}{5}}\\ \beta_1\\ \beta_{\frac{6}{5}}\\ \lambda_0\\ \lambda_{\frac{4}{5}}\\ \lambda_1\\ \lambda_{\frac{6}{5}}\\ \delta_2\end{pmatrix};$$

$$B=\begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
0 & 2 & -8 & 18 & -32 & 50 & -72 & 98 & -128 & 162\\
0 & 2 & \frac{24}{5} & \frac{66}{25} & \frac{-672}{125} & \frac{-1558}{125} & \frac{-30888}{3125} & \frac{56434}{15625} & \frac{1416576}{78125} & \frac{7747938}{390625}\\
0 & 2 & 8 & 18 & 32 & 50 & 72 & 98 & 128 & 162\\
0 & 2 & \frac{56}{5} & \frac{1026}{25} & \frac{16352}{125} & \frac{48682}{125} & \frac{3476088}{3125} & \frac{48256754}{15625} & \frac{656238464}{78125} & \frac{8784645858}{390625}\\
0 & 0 & 16 & -96 & 320 & -800 & 1680 & -3136 & 5376 & -8640\\
0 & 0 & 16 & \frac{288}{5} & \frac{1856}{25} & \frac{-288}{25} & \frac{-23472}{125} & \frac{-915264}{3125} & \frac{2106112}{15625} & \frac{21577536}{78125}\\
0 & 0 & 16 & 96 & 320 & 800 & 1680 & 3136 & 5376 & 8640\\
0 & 0 & 16 & \frac{672}{5} & \frac{17216}{25} & \frac{71008}{25} & \frac{1297488}{125} & \frac{109756864}{3125} & \frac{1760464128}{15625} & \frac{27152483904}{78125}\\
0 & 0 & 0 & 192 & 4608 & 68160 & 801792 & 8227968 & 77064192 & 675946368
\end{pmatrix};$$

and

$$C=\begin{pmatrix}t^0\\ t^1\\ t^2\\ t^3\\ t^4\\ t^5\\ t^6\\ t^7\\ t^8\\ t^9\end{pmatrix}$$

Equation (2.6) is evaluated at $x = x_{n+1}(t = 1)$,$x = x_{n+\frac{4}{5}}(t = \frac{3}{5})$,$x = x_{n+\frac{6}{5}}(t = \frac{7}{5})$ and $x = x_{n+2}(t = 3)$. This produces the following schemes

(2.7)

$$\begin{pmatrix} y_{n+\frac{4}{5}} \\ y_{n+1} \\ y_{n+\frac{6}{5}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} y_n + hD \begin{pmatrix} f_n \\ f_{n+\frac{4}{5}} \\ f_{n+1} \\ f_{n+\frac{6}{5}} \\ f_{n+2} \end{pmatrix} + h^2 E \begin{pmatrix} g_n \\ g_{n+\frac{4}{5}} \\ g_{n+1} \\ g_{n+\frac{6}{5}} \\ g_{n+2} \end{pmatrix} + h^3 F \begin{pmatrix} X_n \\ X_{n+\frac{4}{5}} \\ X_{n+1} \\ X_{n+\frac{6}{5}} \\ X_{n+2} \end{pmatrix}$$

where

$$D = \begin{pmatrix} \frac{3741217}{20314800} & \frac{239909}{1269675} & \frac{16227}{83600} & \frac{640198}{1269675} \\ \frac{4450547}{21067200} & \frac{147286}{329175} & \frac{1151739}{2340800} & \frac{807392}{329175} \\ \frac{427723}{5266800} & \frac{108446}{329175} & \frac{337851}{585200} & \frac{175312}{329175} \\ \frac{13294271}{568814400} & \frac{303598}{8887725} & \frac{553701}{2340800} & \frac{-13238944}{8887725} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} \frac{61118924}{4041140625} & \frac{312941}{20690640} & \frac{3018459}{199562500} & \frac{345583}{5172660} \\ \frac{-11054516}{5028975} & \frac{-113072125}{51496704} & \frac{-4905927}{2235100} & \frac{23973625}{804636} \\ \frac{-2787991552}{628621875} & \frac{-7146695}{1609272} & \frac{-309668832}{69846875} & \frac{22788256}{201159} \\ \frac{-31234432}{45260775} & \frac{-160001125}{231735168} & \frac{-77388}{111755} & \frac{56805500}{1810431} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{62656}{3143109375} & \frac{2567}{128741760} & \frac{6966}{349234375} & \frac{2798}{1005795} \end{pmatrix}$$

3. Analysis of the Method

The basic features of this approach, such as order, error constant, zero stability, and consistency, are investigated in this section.

3.1. **Order.** Equation (2.7) derived is a discrete scheme belonging to the class of LMMs of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \left(\sum_{j=0}^{k} \beta_j(t) f_{n+j} \right) + h^2 \left(\sum_{j=0}^{k} \lambda_j(t) g_{n+j} \right) + h^3 \left(\sum_{j=0}^{k} \delta_j(t) X_{n+j} \right) \tag{3.1}$$

Following [8] Fatunla (1988), we define the local truncation error associated with equation (3.2) by the difference operator

$$L[y(x) : h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h^2 \beta_j f(x_n + jh) - h^3 \gamma_j g(x_n + jh) \right] \tag{3.2}$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].

Expanding (3.2) using Taylor series about the point$x$, the expression below is obtained

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \ldots + C_{p+2} h^{p+2} y^{p+2}(x)$$

where the C_0, C_1,$C_2 \ldots C_p \ldots C_{p+1}$ are obtained as $C_0 = \sum_{r=0}^{s} \alpha_r$,$C_1 = \sum_{r=1}^{s} r\alpha_r$,$C_2 = \frac{1}{2!} \sum_{r=1}^{s} r^2 \alpha_r$,$C_q = \frac{1}{q!} \left[\sum_{r=1}^{s} r^q \alpha_r - q(q-1) \sum_{r=1}^{s} \beta_r r^{q-2} - q(q-1)(q-2) \sum_{r=1}^{s} \gamma_r r^{q-3} \right]$. In the spirit of [13]), equation (3.2) is of order p if $C_0 = C_1 = C_2 = \ldots C_p = C_{p+1} = 0$and $C_{p+r} \neq 0$. The $C_{p+r} \neq 0$ is called the error constant and$C_{p+r} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point x_n.

Thus, the block (2.7) is of order $p = 8$ and error constant

$$C_{p+2} = \left[\frac{509556736}{27845984619140625}, \frac{33399881}{1824914448000000}, \frac{2797001}{152790039062500}, \frac{3963577}{7128572062500} \right]^T.$$

3.2. **Zero Stability of the Method.** The linear multistep method (3.1) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation. To analyze the zero-stability of the method, we present (3.2) in vector notation form of column vectors $e = (e_1 \dots e_r)^T$, $d = (d_1 \dots d_r)^T$, $y_m = (y_{n+1} \dots y_{n+r})^T$, $F(y_m) = (f_{n+1} \dots f_{n+r})^T$,$G(y_m) = (g_{n+1} \dots g_{n+r})^T$, $W(y_m) = (T_{n+1} \dots T_{n+r})^T$and matrices $A = (a_{ij})$, $B = (b_{ij})$.
Thus, equation (3.2) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hbf_n + h^2 DG(y_m) + h^2 dg_n + h^3 VW(y_m) + h^3 uX_n \tag{3.3}$$

where h is a fixed mesh size within a block.
In line with (3.3),

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; b = \begin{pmatrix} \frac{2798861662}{12123421875} \\ \frac{-23699458}{1005795} \\ \frac{38973022208}{3143109375} \\ \frac{318614176}{27156465} \\ 0 \end{pmatrix}; d = \begin{pmatrix} \frac{61118924}{4041140625} \\ \frac{-11054516}{5028975} \\ \frac{-2787991552}{628621875} \\ \frac{-31234432}{45260775} \\ 0 \end{pmatrix};$$

$$v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{62656}{3143109375} \end{pmatrix}; B = \begin{pmatrix} \frac{917148937}{3972602880} & \frac{92147253}{399125000} & \frac{18645763}{31035960} \\ \frac{-4835550625}{205986816} & \frac{-20976327}{894040} & \frac{690888125}{1609272} \\ \frac{12577412}{1005795} & \frac{4403973888}{349234375} & \frac{12446464}{1005795} \\ \frac{8161180625}{27156465} & \frac{1321074}{111755} & \frac{-2391358750}{8887725} \\ 0 & 0 & 0 \end{pmatrix};$$

$$D = \begin{pmatrix} \frac{312941}{20690640} & \frac{3018459}{199562500} & \frac{345583}{5172660} \\ \frac{-113072125}{51496704} & \frac{-4905927}{2235100} & \frac{23973625}{804636} \\ \frac{-7146695}{1609272} & \frac{-309668832}{69846875} & \frac{22788256}{201159} \\ \frac{-160001125}{231735168} & \frac{-77388}{111755} & \frac{56805500}{1810431} \\ 0 & 0 & 0 \end{pmatrix}; \text{and } V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2567}{128741760} & \frac{6966}{349234375} & \frac{2798}{1005795} \end{pmatrix}$$

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = \det(RA^0 - A^1) \tag{3.4}$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{and } A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

substituting A^0 and A^1 in equation (3.4) and solving for R, the values of R are obtained as $0, 0, 0$ and 1.
According to [13], the block method equation (2.7) are zero-stable, since from (3.4), $\rho(R) = 0$, satisfy $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

3.3. **Consistency and convergence of the method.** The linear multistep method in equation (3.1) is said to be consistent if it has order $p \geq 1$. Equation (2.7) is of order 8. According to the theorem [7], the necessary and sufficient condition for a linear multistep method to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

3.4. **Numerical Experiments.** Here in this section, different types of differential problems which include first, second and third-order ordinary differential equations are examined to test the effectiveness of the new scheme.

Problem 1: Consider the stiff problem,

$$y_1^{'} = -y_1; \quad y_1(0) = 1 \quad h = 0.1, \quad 0 \leq x \leq 1$$
$$y_2^{'} = -2000y_2; \quad y_2(0) = 1$$
$$y_1(x) = e^{-x}, \quad y_2(x) = e^{-2000x}$$

Table 1: Comparing the error of the new block method with existing methods for solving Problem 1

x- values	Error in new method	Error in new method	Error in [19]	Error in [19]
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	-1.55E-018	-7.60E-004	6.00E-010	9.52E-001
0.2	-2.81E-018	-5.77E-007	9.00E-010	9.10E-001
0.3	-3.82E-018	-7.84E-013	1.10E-009	8.62E-001
0.4	-4.61E-018	-5.95E-009	1.40E-009	8.20E-001
0.5	-5.21E-018	-8.08E-006	1.50E-009	7.80E-001
0.6	-5.66E-018	-6.14E-009	1.70E-009	7.42E-001
0.7	-5.97E-018	-4.67E-012	1.80E-009	7.10E-001
0.8	-6.18E-018	-4.13E-015	1.80E-009	6.72E-001
0.9	-6.29E-018	-3.14E-018	1.80E-009	6.39E-001
1.0	-6.32E-021	-2.38E-021	1.80E-009	6.08E-001

Problem 2: Consider the stiff problem,

$$y^{'''} = 3\sin x; \; y(0) = 1 \; y^{'}(0) = 0 \; y^{''}(0) = -2 \; h = 0.1$$
$$Exact\ Solution : y(x) = 3\cos x + \tfrac{x^2}{2} - 2$$

Table 2: the error of the new block method with existing methods for solving Problem 2

x- values	Error in the new method	Error in [4]	Error in [12]
0.1	5.4694748E-018	2.000000E-010	6.370460E-13
0.2	1.0864706E-017	4.000000E-010	4.052980E-12
0.3	1.6131788E-017	2.000000E-010	1.00932 6E-11
0.4	2.1218091E-017	2.000000E-010	1.890366E-11
0.5	2.6072798E-017	9.000000E-010	3.033807E-11
0.6	3.0647399E-017	1.100000E-009	4.455258E-11
0.7	3.4896188E-017	1.500000E-009	5.987466E-11
0.8	3.8776710E-017	1.300000E-009	7.711903E-11
0.9	4.2250197E-017	1.500000E-009	9.618412E-11
1.0	4.5281938E-017	2.000000E-009	1.171654E-10

Problem 3:

$$y'' = y';\ y(0) = 0\ y'(0) = -1\ h = 0.1$$
$$Exact\ Solution : y(x) = 1 - e^x$$

Table 3: Comparing the error of the new block method with existing methods for solving Problem 3

x- values	Error in the new method	Error in [3]	Error in [12]
0.1	4.600000E-025	2.095826E-010	2.508826E-13
0.2	8.000000E-026	2.092718E-009	6.493175E-11
0.3	5.000000E-026	7.842546E-009	1.683146E-09
0.4	2.500000E-025	2.009500E-008	1.700635E-08
0.5	3.700000E-025	4.199771E-008	1.025454E-07
0.6	1.300000E-025	7.728842E-008	2.558711E-06
0.7	6.000000E-026	1.303844E-007	5.273300E-06
0.8	2.500000E-025	2.064839E-007	8.275935E-06
0.9	3.800000E-025	3.116817E-007	1.161667E-05
1.0	2.000000E-025	4.531001E-007	1.542187E-05

Problem 4:

$$y''' = e^x;\ y(0) = 3;\ y'(0) = 1;\ y''(0) = 5;\ h = 0.1$$
$$Exact\ Solution : y(x) = 2 + 2x^2 + e^x$$

Table 4: Comparing the error of the new block method with existing methods for solving Problem 4

x- values	Error in the new method	Error in [3]	Error in [12]
0.1	1.994384E-018	8.881784E-015	3.369305E-12
0.2	4.198520E-018	3.552714E-014	2.160050E-11
0.3	6.634467E-018	8.304468E-014	5.333245E-11
0.4	9.326604E-018	1.527667E-013	9.988632E-11
0.5	1.230188E-017	2.460254E-013	1.598988E-10
0.6	1.559006E-017	3.668177E-013	2.511404E-10
0.7	1.922407E-017	5.178080E-013	3.961489E-10
0.8	2.324026E-017	7.025491E-013	5.926823E-10
0.9	2.767885E-017	9.254819E-013	8.429168E-10
1.0	3.258424E-017	1.187495E-012	1.144603E-09

Problem 5:

$$y''' = y'(2xy'' + y'); \ y(0) = 1; \ y'(0) = \tfrac{1}{2}; \ y''(0) = 0; \ h = 0.01$$
$$Exact\ Solution : y(x) = 1 + \tfrac{1}{2}\ln\tfrac{2+x}{2-x}$$

Table 5: Comparing the error of the new block method with existing methods for solving Problem 5

x- values	Error in the new method,	Error in [9]	Error in [16]
0.1	0	1.194048000E-013	2.508826E-13
0.2	0	4.086842000E-013	6.493175E-11
0.3	1.0E-024	1.016689500E-012	1.683146E-09
0.4	0	2.139483600E-012	1.700635E-08
0.5	0	4.083580200E-012	1.025454E-07
0.6	1.0E-024	7.350069300E-012	2.558711E-06
0.7	0	1.279204250E-012	5.273300E-06

Discussion of Results. Tables I-V above show the tabular display of the numerical solutions on the implementation of the newly developed method. It is evident that the block method is more efficient in terms of accuracy when compared with existing methods.

Conclusion: The derivation of a new block approach for directly solving first, second, and third-order ordinary differential equations is investigated in this work. The method's ability to solve three distinct orders of ODEs successfully is one of its most appealing characteristics. The novel approach is applied to several differential equations of order one, two, and three to demonstrate its efficiency. The solutions of test problems solved outperform the existing methods in terms of accuracy, as shown in Tables I-V.

Acknowledgement: The authors would like to extend their sincere thanks to all that have contributed in one way or the other for the actualization of the review of this work.

Competing interests: The content of the manuscript was approved by all authors. Therefore, no competing interest between authors.

Funding: Authors received no financial support for the research, authorship, and/or publication of this article.

References

[1] Adeniyi R. B. & Alabi M. O. (2011). A collocation method for direct numerical integration of initial value problems in higher-order ordinary differential equations. *Analele Stiintifice Ale Universitatii AL. I. Cuza Din Iasi (SN), Matematica, Tomul.* **2** (1), 311-321.

[2] Adeyefa, E. O. (2017). Orthogonal-based hybrid block method for solving general second-order initial value problems. *Italian Journal of Pure and Applied Math.* **37** (1), 659-672.

[3] Adeyefa, E. O. & Kuboye, J. O. (2020). Derivation of new numerical model capable of solving second and third order ordinary differential equations directly. *IAENG International Journal of Applied Mathematics.* **50** 2, 233-241.

[4] Adeyefa E. O., Olajide O. A., Akinola L. S., Abolarin O. E., Ibrahim A. A. & Haruna Y. (2020). On direct integration of second and third order ODES. *J. Eng. and Applied Sci.* **15** (8),1972-1976.

[5] Ajileye G., Amoo S. A. & Ogwumu O. D. (2018). Hybrid block method algorithms for solution of first order initial value problems in ordinary differential equations. *Journal of Applied and Computational Math.* **7** (2), 1-4.

[6] Awoyemi D. O. (1999). A class of continuous linear multistep methods for general second-order initial value problems in ordinary differential equations. *Intern. J. Computer Math.* **72** (2), 29-37.

[7] Dahlquist G. (1979). *Some properties of linear multistep and one leg method for ordinary differential equations*. Department of computer science, Royal institute of technology, Stockholm.
[8] Fatunla S. O. (1991), Block method for second-order initial value problem (IVP). *International Journal of Computer Math.* **41** (9), 55-63.
[9] Folaranmi R. O., Adeniyi R. B. & Adeyefa E. O. (2016). An orthogonal Based Self-Starting Numerical Integrator for third order IVPs in ODEs. *Pacific Journal of Science and Technology.* **15**, 126-132.
[10] Ismail F., Ken Y. L. & Othman M. (2009). Explicit and Implicit 3-point block methods for solving special second order ordinary differential equations directly. *Intern. J. Math. Anal.* **18** (2), 239-254.
[11] Kayode S. J. & Adegboro J. O. (2018). Predator-corrector Linear Multistep Method for Direct Solution of Initial Value Problems of Second Order Ordinary Differential Equations. *Asian J. Physical and Chemical Sci.* **6** (1), 1-9.
[12] Kuboye J. O. (2015). *Block methods for direct solution of higher-order ordinary differential equations using interpolation and collocation approach*. Ph.D. thesis, Universiti Utara Malaysia, (2015).
[13] Lambert J. D. (1973). *Computational methods for ordinary differential equations*. John Wiley and Sons.
[14] Lambert J. D. (1991). *Numerical methods for ordinary differential systems*. John Wiley and Sons.
[15] Mohammed U. & Adeniyi R. B. (2014). Derivation of five-step Block Hybrid Backward Differential Formulas (HBDF) through the continuous multi-step collocation for solving second order differential equation. *Pacific J. Sci. and Tech.* **15** (2), 89-95.
[16] Olabode B. T. (2009). An accurate scheme by block method for the third-order ordinary differential equation. *Pacific J. Sci. and Tech.* **10** (1), 136-142.
[17] Olabode B. T. (2013). Block multistep method for the direct solution of the third order of ordinary differential equations. *FUTA J. Research in Sci.* **9** (2), 194-200.
[18] Ramos H., Mehta S. & Vigo-Aguiar J. A. (2016). Unified approach for the development of k-Step block Falkner-type methods for solving general second-order initial-value problems in ODEs. *J. Computational and Applied Math.* article 12. 12-18.
[19] Sabo J., Kyagya T. Y. & Ayinde A. M. (2020). The formation of implicit second order backward difference Adam's formulae for solving stiff systems of first order initial value problems of ordinary differential equations. *Asian J. Advan. Research and Rep.* **10** (4), 21-29.