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A new collocation approach for a numerical treatment of multi order fractional differential equations

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Abstract

This paper discusses the numerical solution of multi order fractional differential equations by Exponentially fitted collocation method using two basis functions namely: Bernoulli polynomial and shifted Legendre polynomials.The fractional derivatives are expressed in Caputo sense. The assumed solution is substituted into the problem considered an then manipulated. The resulting equation is then collocated at equi-distance points. Hence, reducing it to algebraic linear system of equations which are then solved using Maple 18 to obtain the unknown constants. Three numerical examples are considered to test the applicability and accuracy of the method. The results obtained show that the trial solution with Bernoulli polynomial performed well than shifted Legendre polynomials in the problem considered.

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1. Introduction and Preliminaries

Multi-order fractional differential equations(MFDE) have been found to be effective in describing the mathematical phenomenon such as damping laws, diffusion processes, physics, chemistry, viscoelastic materials, polymers, and so on. Some Multi-order fractional differential equations(MFDE) do not have exact analytical solutions. In recent years, numerous approaches have been studied for the numerical solution of multi order fractional differential equations such as operational matrix [1], spectral collocation methods [2], stable fractional chebyshev differentiation matrix [3] and fractional operational methods [6]. [7] used sine-cosine wavelet operational matrix of fractional integrations and its applications in solving the fractional order riccati differential equations. On the other hand, [4] used generalized triangular function operational matrices to solve multi order fractional differential equations(M-FDEs), while [5] used Chebyshev wavelets to solve non linear fractional differential equations(FDEs). In this work, exponentially fitted collocation method is used as a numerical tools for multi order fractional differential equations. The methods considered reduced the problem under consideration to algebraic system of equations by using Bernoulli polynomials and shifted Legendre polynomials as basis functions. The comparison of the two basis are made to show the best in accuracy, efficiency and simplicity.

The arrangement of the paper goes thus, section 2 discussed relevant terms, in section 3 demonstrate the methodology, section 4 give the numerical examples solution, section 5 are tables of results while in section 6 discussed the conclusion of the work.

2. Relevant terms

Fractional differential equations(FDE). Fractional differential equations are generalization of classical differential equations of several different possibilities of defining any arbitrary order, real number powers or complex numbers powers of the differentiation operator D.
(Katugampola and Udita, N. (2014)).

Legendre polynomial. Legendre polynomial is defined by the Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \tag{1}$$

It is observed from the definition that the n^{th} derivative must be carried out before a polynomial of degree n are obtained. Thus, the first few set of Legendre

Polynomials are obtained as follows:

$$
\begin{aligned}
&P_0(x) = 1\\
&P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}[(x^2-1)] = x\\
&P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}[(x^2-1)^2] = \frac{1}{2}(3x^2-1)\\
&\vdots
\end{aligned} \tag{2}
$$

As n increases, differentiation of equation (2) becomes more tedious. An easy way of generating the terms of Legendre Polynomials is given by its recurrence relation below

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right) xP_n(x) - \frac{n}{n+1}P_{n-1}(x),\ n \geq 1 \tag{3}$$

The shifted Legendre Polynomials is define on the interval $[a, b]$. It is necessary to shift the defining domain by means of the following substitution

$$X = \frac{2x-a-b}{b-a};\ a \leq X \leq b \tag{4}$$

The shifted Legendre Polynomials in x are then obtained by substituting equation (4) into the Rodrigue's and recurrence relation formula as follows

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(\left(\frac{2x-a-b}{b-a}\right)^2 - 1\right)^n\right] \tag{5}$$

and

$$P^*_{n+1}(x) = \left(\frac{(2n+1)(2x-a-b)}{(n+1)(b-a)}\right) P^*_n(x) - \frac{n}{n+1}P^*_{n-1}(x),\ n \geq 1 \tag{6}$$

The first few shifted Legendre Polynomials are given as

$$
\begin{aligned}
P^*_0(x) &= 1\\
P^*_1(x) &= 2x-1\\
P^*_2(t) &= 6x^2-6x+1\\
P^*_3(x) &= 20x^3-30x^2+12x-1\\
P^*_4(x) &= 70x^4-140x^3+90x^2-20x+1\\
P^*_5(x) &= 252x^5-630x^4+560x^3-210x^2+30x-1\\
&\vdots
\end{aligned} \tag{7}
$$

Chebyshev polynomials. The Chebyshev polynomial $T_n(x)$ can be defined by means of Rodrigue's formula

$$T_n^*(x) = \frac{(-2)^2 n!}{(2n)!}\sqrt{1-x^2}\frac{d^n}{dx^n}(1-x^2)^n - \frac{1}{2} \qquad n = 0, 1, 2, \cdots \tag{8}$$

The first few terms are obtained by substituting $n = 0, 1, 2, \cdots$ in equation (8). Thus, the first few terms are as follows:

$$\begin{aligned} T_0^*(x) &= 1 \\ T_1^*(x) &= x \\ T_2^*(x) &= 2x^2 - 1 \\ &\vdots \end{aligned}$$

As n increases, differentiation of equation (8) becomes more cubersome. An easy way of generating the other terms since the first two terms are known and can be obtained by means of the recurrence relation

$$T_{n+1}^*(x) = 2xT_n^*(x) - T_{n-1}^*(x) \tag{9}$$

The shifted Chebyshev Polynomials is define on the interval $[0, 1]$ and is defined by

$$T_n(x) = T_n(2x - 1) \tag{10}$$

The term terms of shifted Chebyshev polynomial are obtained below

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 2x - 1 \\ T_2(x) &= 8x^2 - 8x + 1 \\ T_3(x) &= 32x^3 - 48x^2 + 18x - 1 \\ T_4(x) &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ &\vdots \end{aligned}$$

Bernoulli number. A generating function approach is a convenient way to introduce the set of numbers firstly used in mathematics by Jacob Bernoulli. Bernoulli numbers is denoted by B_n and its generating function formula is given by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} \tag{11}$$

Since equation (11) is a Taylor series expansion and B_n is identify as successive derivatives of the generating function

$$B_n = \left[\frac{d^n}{dt^n}\left(\frac{t}{e^t-1}\right)\right]_{t=0} \qquad n = 0, 1, 2, \cdots \tag{12}$$

When $n = 0$ in equation (12), we have

$$B_0 = \left[\left(\frac{t}{e^t-1}\right)\right]_{t=0} \tag{13}$$

Taking the limit of equation (13), we obtain

$$B_0 = 1$$

Therefore when $n = 1, 2, 3, \cdots$ in (12) , we obtain the values of B_n as follows

$$\begin{aligned} B_1 &= -\frac{1}{2} \\ B_2 &= \frac{1}{6} \\ B_3 &= 0 \\ &\vdots \end{aligned}$$

Bernoulli polynomials. Bernoulli polynomials is denoted by $B_n(x)$ and its generating function formula is given by

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} \tag{14}$$

Where $B_n(x)$ is a Bernoulli polynomials. The recursive relation formula is given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}x^k \qquad n \geq 0 \tag{15}$$

The first few terms of Bernoulli polynomials are given below

$$\begin{aligned}
&B_0(x) = 1\\
&B_1(x) = x - \frac{1}{2}\\
&B_2(x) = x^2 - x + \frac{1}{6}\\
&B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\\
&B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}\\
&\vdots
\end{aligned}$$

Caputo fractional derivatives. Suppose that $\alpha > 0, t > \alpha, t \in \mathbb{R}$. The fractional operator

$$D^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_\alpha^t \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & a = n \in \mathbb{N} \end{cases} \tag{16}$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α. The Caputo fractional derivatives of the power function satisfies

$$D^\alpha x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(n-\alpha+1)} x^{p-\alpha}, & n-1 < \alpha < n, p > n-1, p \in \mathbb{N} \\ 0, & n-1 < \alpha < n, p \le n-1, p \in \mathbb{N} \end{cases} \tag{17}$$

3. Methodology

3.1. **Exponentially fitted collocation method.** Consider the general form of multi-order fractional differential equation

$$D^\alpha v(x) + \sum_{m=0}^{n} Y_m D^{\beta_m} v(x) = g(x) \qquad x \in [0,1] \tag{18}$$

together with the initial conditions

$$v^{(j)}(0) = C_j, \qquad j = 0, 1 \tag{19}$$

where $p - 1 < \alpha \le p$, $Y_m(m = 0, 1, 2, \cdots, n)$ is a constants co-efficient, $0 < \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_n < \alpha$, $g(x)$ is a given function and $v(x)$ is the unknown function to be determined.

Now, consider the trial solutions of the following

i. Non exponentially fitted trial solution

$$v_k(x) = \sum_{i=0}^{k} a_i Q_i(x) \tag{20}$$

ii. Exponentially fitted trial solution

$$v_k(x) = \sum_{i=0}^{k} a_i Q_i(x) + \tau_k e^x \tag{21}$$

Where $Q_i(x)$ $(i = 0, 1, 2, \cdots, k)$ is the choosen basis functions. Substituting equation (20) into equation (18) to obtain

$$D^\alpha \left(\sum_{i=0}^{k} a_i Q_i(x) \right) + \sum_{m=0}^{n} Y_m D^{\beta_m} \left(\sum_{i=0}^{k} a_i Q_i(x) \right) = g(x) \tag{22}$$

Equation (22) is re written as

$$\sum_{i=0}^{k} \left(D^\alpha \left(a_i Q_i(x) \right) + \sum_{m=0}^{n} Y_m D^{\beta_m} \left(a_i Q_i(x) \right) \right) = g(x) \tag{23}$$

Equation (23) is slightly perturb, thus

$$\sum_{i=0}^{k} \left(D^\alpha \left(a_i Q_i(x) \right) + \sum_{m=0}^{n} Y_m D^{\beta_m} \left(a_i Q_i(x) \right) \right) - W_k(x) = g(x) \tag{24}$$

Where $W_k(x)$ is perturbation parameters, defined as

$$W_k(x) = \sum_{i=1}^{\alpha} \tau_i T_{k-i+1}(x)$$

and τ_i are free tau parameter to be determined. $T_{k-i+1}(x)$ are Shifted Chebyshev polynomial of the first kind defined in equation (10). Now, collocating equation (24) at an equi-distance points $x = x_l$, to have

$$\sum_{i=0}^{k} \left(D^\alpha \left(a_i Q_i(x_l) \right) + \sum_{m=0}^{n} Y_m D^{\beta_m} \left(a_i Q_i(x_l) \right) \right) - W_k(x_l) = g(x_l) \tag{25}$$

Where

$$x_l = a + \left(\frac{b-a}{k+2} \right) l \qquad l = 1, 2, 3, \cdots, k+1 \tag{26}$$

Hence, equation (25) gives rise to system of linear algebraic equations in $(k+2)$ unknown constants. Two additional equations obtained by plugging our trial solutions into the equation (21) are

$$v_k(x_0) \Rightarrow \sum_{i=0}^{k} a_i Q_i(x_0) + \tau_k e^{x_0} = C_0 \tag{27}$$

and

$$\frac{d}{dx}v_k(x_0) \Rightarrow \sum_{i=0}^{k} \frac{d}{dx}\left(a_i Q_i(x_0)\right) + \frac{d}{dx}\left(\tau_k e^{x_0}\right) = \frac{d}{dx}C_0 \tag{28}$$

Thus, $(k+2)$ by $(k+2)$ algebraic linear system of equations are obtained and solved using **MAPLE 18** software and obtained the unknown constants a_i $(i = 0, 1, 2, \cdots, k)$ which are then substituted into equation (21) to obtain the approximate solution.

4. Numerical examples

In this Chapter, an illustrative demonstration of the method discussed above is carried out on three examples of multi-order fractional differential equations.

4.1. **Example 1.** Consider the following multi-order fractional differential equation

$$D^{\alpha}v(x) + \sum_{m=0}^{n} Y_m D^{\beta_m} v(x) = g(x) \tag{29}$$

with the initial conditions

$$v^{(0)}(0) = C_0, \qquad v^{(1)}(0) = C_1 \tag{30}$$

where $n = 3$, $\alpha = 2$, $C_0 = C_1 = 0$,$Y_0 = Y_2 = 1$,$Y_1 = -2$,$Y_3 = 0$,$\beta_0 = 0$,$\beta_1 = 1$,$\beta_2 = \frac{1}{2} \in (0, 1)$ and the given function is

$$g(x) = x^7 + \frac{2048}{429\sqrt{\pi}}x^{6.5} - 14x^6 + 42x^5 - x^2 - \frac{8}{3\sqrt{\pi}}x^{1.5} + 4x - 2 \tag{31}$$

the exact solution is

$$v(x) = x^7 - x^2 \tag{32}$$

Shifted Legendre polynomial basis function. Here, equation (29) is solved for case $k = 8$. Thus, the trial solution given in equation (20) and equation (21) are

i. Trial solution

$$v_k(x) = \sum_{i=0}^{k} a_i P_i(x) \tag{33}$$

ii. Exponentially fitted trial solution

$$v_k(x) = \sum_{i=0}^{k} a_i P_i(x) + \tau_k e^{x} \tag{34}$$

Expanding equation (33) gives

$$
\begin{aligned}
v_8(x) =& a_0 + a_1\left(2\,x-1\right) + a_2\left(6\,x^2-6\,x+1\right) + a_3\left(20\,x^3-30\,x^2+12\,x-1\right)\\
&+ a_4\left(70\,x^4-140\,x^3+90\,x^2-20\,x+1\right)+\\
&a_5\left(252\,x^5-630\,x^4+560\,x^3-210\,x^2+30\,x-1\right)+\\
&a_6\left(924\,x^6-2772\,x^5+3150\,x^4-1680\,x^3+420\,x^2-42\,x+1\right)\\
&+ a_7\left(3432\,x^7-12012\,x^6+16632\,x^5-11550\,x^4+4200\,x^3-756\,x^2+56\,x-1\right)+\\
&a_8\left(12870\,x^8-51480\,x^7+84084\,x^6-72072\,x^5+34650\,x^4-9240\,x^3+1260\,x^2-72\,x+1\right)
\end{aligned} \tag{35}
$$

Substituting equation (35) into equation (29) and simplify to obtain

$$
\begin{aligned}
&-2\,a_8\left(102960\,x^7-360360\,x^6+432432\,x^5+138600\,x^3-27720\,x^2+2520\,x-72\right)\\
&-2\,a_2\left(12\,x-6\right)-2\,a_3\left(60\,x^2-60\,x+12\right)-2\,a_4\left(280\,x^3-420\,x^2+180\,x-20\right)\\
&-2\,a_5\left(1260\,x^4-2520\,x^3+1680\,x^2-420\,x+30\right)-2\,a_6\Big(5544\,x^5-13860\,x^4\\
&+12600\,x^3-5040\,x^2+840\,x-42\Big)-2\,a_7\Big(24024\,x^6-72072\,x^5+83160\,x^4-46200\,x^3\\
&+12600\,x^2-1512\,x+56\Big)+a_8\Big(720720\,x^6-2162160\,x^5+2522520\,x^4-1441440\,x^3\\
&+415800\,x^2-55440\,x+2520\Big)+a_7\Big(144144\,x^5-360360\,x^4+332640\,x^3-138600\,x^2\\
&+25200\,x-1512\Big)+a_5\left(5040\,x^3-7560\,x^2+3360\,x-420\right)+a_6\Big(27720\,x^4-55440\,x^3\\
&+37800\,x^2-10080\,x+840\Big)+a_3\left(120\,x-60\right)+a_4\left(840\,x^2-840\,x+180\right)\\
&-\tau_1\Big(32768\,x^8-131072\,x^7+212992\,x^6-180224\,x^5-84480\,x^4-21504\,x^3+2688\,x^2\\
&-128\,x+1\Big)-\tau_2\Big(8172\,x^7-28672\,x^6+39424\,x^5-26880\,x^4+9408\,x^3-1568\,x^2\\
&+98\,x-1\Big)+a_8\Big(65536\,\frac{x^{15/2}}{\sqrt{\pi}}-245760\,\frac{x^{13/2}}{\sqrt{\pi}}+372736\,\frac{x^{11/2}}{\sqrt{\pi}}-292864\,\frac{x^{9/2}}{\sqrt{\pi}}\\
&+126720\,\frac{x^{7/2}}{\sqrt{\pi}}-29568\,\frac{x^{5/2}}{\sqrt{\pi}}+3360\,\frac{x^{3/2}}{\sqrt{\pi}}-144\,\frac{\sqrt{x}}{\sqrt{\pi}}\Big)
\end{aligned}
$$

$$+ a_7\Big(16384\,\frac{x^{13/2}}{\sqrt{\pi}} - 53248\,\frac{x^{11/2}}{\sqrt{\pi}} + 67584\,\frac{x^{9/2}}{\sqrt{\pi}} - 42240\,\frac{x^{7/2}}{\sqrt{\pi}} + 13440\,\frac{x^{5/2}}{\sqrt{\pi}} - 2016\,\frac{x^{3/2}}{\sqrt{\pi}}$$
$$+ 112\,\frac{\sqrt{x}}{\sqrt{\pi}}\Big) + a_6\Big(4096\,\frac{x^{11/2}}{\sqrt{\pi}} - 11264\,\frac{x^{9/2}}{\sqrt{\pi}} + 11520\,\frac{x^{7/2}}{\sqrt{\pi}} - 5376\,\frac{x^{5/2}}{\sqrt{\pi}} + 1120\,\frac{x^{3/2}}{\sqrt{\pi}}$$
$$- 84\,\frac{\sqrt{x}}{\sqrt{\pi}}\Big) + a_5\left(1024\,\frac{x^{9/2}}{\sqrt{\pi}} - 2304\,\frac{x^{7/2}}{\sqrt{\pi}} + 1792\,\frac{x^{5/2}}{\sqrt{\pi}} - 560\,\frac{x^{3/2}}{\sqrt{\pi}} + 60\,\frac{\sqrt{x}}{\sqrt{\pi}}\right)$$
$$+ a_4\left(256\,\frac{x^{7/2}}{\sqrt{\pi}} - 448\,\frac{x^{5/2}}{\sqrt{\pi}} + 240\,\frac{x^{3/2}}{\sqrt{\pi}} - 40\,\frac{\sqrt{x}}{\sqrt{\pi}}\right) + a_3\left(64\,\frac{x^{5/2}}{\sqrt{\pi}} - 80\,\frac{x^{3/2}}{\sqrt{\pi}} + 24\,\frac{\sqrt{x}}{\sqrt{\pi}}\right)$$
$$+ a_2\left(16\,\frac{x^{3/2}}{\sqrt{\pi}} - 12\,\frac{\sqrt{x}}{\sqrt{\pi}}\right) + 4\,\frac{a_1\sqrt{x}}{\sqrt{\pi}} + a_1\left(2\,x - 1\right) + a_2\left(6\,x^2 - 6\,x + 1\right)$$
$$+ a_3\left(20\,x^3 - 30\,x^2 + 12\,x - 1\right) + a_4\left(70\,x^4 - 140\,x^3 + 90\,x^2 - 20\,x + 1\right)$$
$$+ a_5\left(252\,x^5 - 630\,x^4 + 560\,x^3 - 210\,x^2 + 30\,x - 1\right) + a_6\Big(924\,x^6 - 2772\,x^5 + 3150\,x^4$$
$$- 1680\,x^3 + 420\,x^2 - 42\,x + 1\Big) + a_0 - 4\,a_1 + 12\,a_2 + a_7\Big(3432\,x^7 - 12012\,x^6 + 16632\,x^5$$
$$- 11550\,x^4 + 4200\,x^3 - 756\,x^2 + 56\,x - 1\Big) + a_8\Big(12870\,x^8 - 51480\,x^7 + 84084\,x^6$$
$$- 72072\,x^5 + 34650\,x^4 - 9240\,x^3 + 1260\,x^2 - 72\,x + 1\Big) = x^7 + \frac{2048\,x^{6.5}}{429\,\sqrt{\pi}}$$
(36)
$$- 14\,x^6 + 42\,x^5 - x^2 - 8/3\,\frac{x^{1.5}}{\sqrt{\pi}} + 4\,x - 2$$

Collocating equation (36) at the point $x_l = a + \left(\frac{b-a}{k+2}\right) l$ where $l = 1, 2, \cdots, k+1$.
When $x_1 = \frac{1}{10}$ in equation (36) gives

$$148.4670753\,a_6 + 20.20451092\,a_2 + 112.3987874\,a_4 - 155.4544727\,a_5 - 58.31121747\,a_3$$
$$- 167.3461527\,a_8 - 62.58307652\,a_7 - 4.086350354\,a_1 + 16.47402752\,\tau_1 - 0.2063852000\,\tau_2$$
(37)
$$+ a_0 = -1.657169691$$

When $x_2 = \frac{2}{10}$ in equation (36) gives

$$269.9140275\,\tau_1 + 0.9787264000\,\tau_2 + 40.32579008\,a_4 - 3.590746991\,a_1 - 1175.608050\,a_8$$
$$- 72.48903945\,a_6 + 115.4438362\,a_7 - 37.77557206\,a_3 + 17.01964338\,a_2 + 0.6889169712\,a_5$$
(38)
$$+ a_0 = -1.361933179$$

When $x_3 = \frac{3}{10}$ in equation (36) gives

$$1369.564690\,\tau_1 - 5391.577875\,a_8 - 21.35321552\,a_7 - 50.31154645\,a_6 + 48.48156533\,a_5$$
$$- 3.163922554\,a_1 - 4.468099261\,a_4 - 20.58004848\,a_3 + 14.31506060\,a_2 - 0.2536484000\,\tau_2$$
(39)
$$+ a_0 = -1.044067350$$

When $x_4 = \frac{4}{10}$ in equation (36) gives

$$4325.416056\,\tau_1 + 11.96178099\,a_2 - 16976.25442\,a_8 - 65.63439537\,a_7 + 23.59278810\,a_6$$
$$+ 33.62800982\,a_5 - 2.772700708\,a_1 - 26.55679532\,a_4 - 6.120712396\,a_3 - 0.9542528000\,\tau_2$$
(40)
$$+ a_0 = -0.5592614403$$

When $x_5 = \frac{5}{10}$ in equation (36) gives

$$- 2.404230879\,a_1 + 9.904230879\,a_2 - 29.62500000\,a_4 + 50.59173088\,a_6 - 40618.97656\,a_8$$
$$+ 6.0\,a_3 - 5.904230879\,a_5 + 8.750000000\,a_7 + 10559.0\,\tau_1 + 0.1562500000\,\tau_2$$
(41)
$$+ a_0 = 0.3493973919$$

When $x_6 = \frac{6}{10}$ in equation (36) gives

$$21897.25606\,\tau_1 - 2.051922511\,a_1 - 16.79504643\,a_4 + 16.10046140\,a_3 + 8.111153507\,a_2$$
$$- 82180.94017\,a_8 + 68.43207036\,a_7 + 9.147831951\,a_6 - 38.38368249\,a_5 + 1.546892800\,\tau_2$$
(42)
$$+ a_0 = 2.078836253$$

When $x_7 = \frac{7}{10}$ in equation (36) gives

$$40568.28469\,\tau_1 - 37.35287295\,a_5 + 1.905108400\,\tau_2 - 1.711860512\,a_1 + 9.254806556\,a_4$$
$$+ 24.45789768\,a_3 + 6.562372102\,a_2 - 148728.6682\,a_8 + 4.897944915\,a_7 - 57.94868040\,a_6$$
(43)
$$+ a_0 = 5.188191956$$

When $x_8 = \frac{8}{10}$ in equation (36) gives

$$69205.59403\,\tau_1 - 54.38735862\,a_6 + 19.14037159\,a_5 - 1.381493983\,a_1 + 46.23112251\,a_4$$
$$+ 31.32444144\,a_3 + 5.243701203\,a_2 - 248059.8275\,a_8 - 111.9001167\,a_7 + 3.215833600\,\tau_2$$
(44)
$$+ a_0 = 10.41723616$$

When $x_9 = \frac{9}{10}$ in equation (36) gives

$$\begin{aligned}&110854.2340\,\tau_1 + 94.00255607\,a_7 + 160.4614035\,a_6 + 149.2148848\,a_5 - 1.059051061\,a_1\\&+ 92.18997927\,a_4 + 36.93531098\,a_3 + 4.144569364\,a_2 - 387806.7503\,a_8 + 9.772325200\,\tau_2\\&+ a_0 = 18.70205307\end{aligned} \tag{45}$$

The other two equations are obtained by plugging equation (30) on equation (34) to have

$$v^{(0)}(0) = 0 \Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 + \tau_2 = 0 \tag{46}$$

$$\begin{aligned}\frac{d}{dx}(v^{(0)}(0)) = 0 \Rightarrow &2\,a_1 - 6\,a_2 + 12\,a_3 - 20\,a_4 + 30\,a_5 - 42\,a_6 + 56\,a_7 - 72\,a_8\\&+ \tau_2 = 0\end{aligned} \tag{47}$$

Equation(37)-(47) are solved using **MAPLE 18** to obtain the unknown constants

$a_0 = -0.2083321282 \quad a_1 = -0.2083306020 \quad a_2 = 0.1250025684 \quad a_3 = 0.1856075395$

$a_4 = 0.07954599941 \quad a_5 = 0.02243601974 \quad a_6 = 0.003787893051 \quad a_7 = 0.0002913758017$

$a_8 = 2.632526388 \times 10^{-12} \quad \tau_1 = -1.203205399 \times 10^{-9} \quad \tau_2 = 3.722041309 \times 10^{-10}$

Putting these constant into equation(34) and after simplification gives

$$\begin{aligned}v_8(x) = &- 6.332454383 \times 10^{-8}\,x^3 + 2.102170393 \times 10^{-7}\,x^4 + 3.388061461 \times 10^{-8}\,x^8\\&- 3.780674736 \times 10^{-10} + 1.000001615\,x^7 + 3.722041309 \times 10^{-10}\,\mathrm{e}^x - 0.9999999948\,x^2\\&+ 8.1045 \times 10^{-10}\,x + 7.2703533 \times 10^{-6}\,x^6 - 4.2173 \times 10^{-7}\,x^5\end{aligned} \tag{48}$$

Bernoulli polynomial basis function. Here, the same procedure is followed here to obtain the below approximate solutions

$$\begin{aligned}v_8(x) = &- 1.05 \times 10^{-8}\,x - 0.9999999086\,x^2 - 5.25 \times 10^{-7}\,x^3 - 1.002024584 \times 10^{-8}\\&+ 1.012226999 \times 10^{-8}\,\mathrm{e}^x + 2.910737515 \times 10^{-8}\,x^8 + 1.000005685\,x^7\\&+ 0.00002574283\,x^6 - 0.000002516\,x^5 + 0.000001596082\,x^4\end{aligned} \tag{49}$$

4.2. **Example 2.** Consider the following multi-order fractional differential equation

$$D^{\alpha}v(x) + \sum_{m=0}^{n} Y_m D^{\beta_m} v(x) = g(x) \tag{50}$$

with the initial conditions

$$v^{(0)}(0) = C_0, \qquad v^{(1)}(0) = C_1 \tag{51}$$

where $n = 3$, $\alpha = 2$, $C_0 = C_1 = 0$,$Y_0 = Y_2 = 1$,$Y_1 = 0$,$Y_3 = -2$,$\beta_0 = 0$,$\beta_2 = \frac{2}{3} \in (0,1)$,$\beta_2 = \frac{5}{3} \in (0,2)$ and the given function is

$$g(x) = x^3 + 6x - \frac{12}{\Gamma(\frac{7}{3})} x^{\frac{4}{3}} + \frac{6}{\Gamma(\frac{10}{3})} x^{\frac{7}{3}} \tag{52}$$

the exact solution is

$$v(x) = x^3 \tag{53}$$

Shifted Legendre polynomial basis function. Here, the same procedure is followed here to obtain the below approximate solutions

$$\begin{aligned} v_8(x) =& 6.30046576 \times 10^{-8}\, x + 0.9999948691\, x^3 + 0.000001485206455\, x^2 + 0.00003090518433\, x^4 \\ &- 0.00007030148745\, x^5 + 0.0001024499786\, x^6 - 0.00007345644428\, x^7 \\ &+ 0.00002492089774\, x^8 - 6.240534227 \times 10^{-8}\, \mathrm{e}^x + 6.230775510 \times 10^{-8} \end{aligned} \tag{54}$$

Bernoulli polynomial basis function. Here, the same procedure is followed here to obtain the below approximate solutions

$$\begin{aligned} v_8(x) =& 8.037672 \times 10^{-9}\, x + 0.9999993189\, x^3 + 2.04994513 \times 10^{-7}\, x^2 + 7.0704509 \times 10^{-9} \\ &- 7.330702650 \times 10^{-9}\, \mathrm{e}^x + 0.000003045687054\, x^8 - 0.000009000068854\, x^7 \\ &+ 0.00001266073474\, x^6 - 0.00000876588769\, x^5 + 0.000003964018814\, x^4 \end{aligned} \tag{55}$$

4.3. **Example 3.** Consider the following multi-order fractional differential equation

$$D^{\alpha} v(x) + \sum_{m=0}^{n} Y_m D^{\beta_m} v(x) = g(x) \tag{56}$$

with the initial conditions

$$v^{(0)}(0) = C_0, \qquad v^{(1)}(0) = C_1 \tag{57}$$

where $n = 3$, $\alpha = 2$, $C_0 = C_1 = 0$,$Y_0 = Y_1 = 1$,$Y_2 = 0$,$Y_3 = -2$,$\beta_0 = 0$,$\beta_1 = \frac{1}{2} \in (0,1)$,$\beta_2 = 1$ and the given function is

$$g(x) = x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}} x^{\frac{5}{2}} \tag{58}$$

the exact solution is

$$v(x) = x^3 \tag{59}$$

Shifted Legendre polynomial basis function. Here, the same procedure is followed here to obtain the below approximate solutions

$$\begin{aligned} v_8(x) =& 1.660760359 \times 10^{-9}\, x + 1.00000024\, x^3 - 3.7789275 \times 10^{-8}\, x^2 - 8.1297065 \times 10^{-7}\, x^4 \\ &+ 1.480558244 \times 10^{-6}\, x^5 - 0.00001285946952\, x^6 - 2.305672281 \times 10^{-6}\, x^7 \\ &- 1.45245 \times 10^{-7}\, x^8 - 2.023363 \times 10^{-9}\, \mathrm{e}^x + 2.12466 \times 10^{-9} \end{aligned} \tag{60}$$

Bernoulli polynomial basis function. Here, the same procedure is followed here to obtain the below approximate solutions

$$\begin{aligned} v_8(x) =& - 2.4319685 \times 10^{-9}\, x + 0.999999824\, x^3 + 2.6187553 \times 10^{-8}\, x^2 + 3.24343648 \times 10^{-9}\, \mathrm{e}^x \\ &- 1.507713165 \times 10^{-8}\, x^8 + 1.755244648 \times 10^{-6}\, x^7 + 7.346184678 \times 10^{-6}\, x^6 - \\ &3.2265879 \times 10^{-7}\, x^5 + 7.170622 \times 10^{-7}\, x^4 - 3.356673 \times 10^{-9} \end{aligned} \tag{61}$$

5. Tables of Results

5.1: Approximate and error solution of example 1

x	exact	Shifted Legendre	Bernoulli Polynomial	Bernoulli Polynomial Error	Shifted Legendre Error
0	0	1.411751×10^{-10}	-3.455038×10^{-10}	3.45×10^{-10}	1.41×10^{-10}
0.1	-0.0099999	-0.009999899	-0.009999900	2.79×10^{-10}	1.54×10^{-10}
0.2	-0.0399872	-0.039987199	-0.039987200	7.00×10^{-11}	2.02×10^{-10}
0.3	-0.0897813	-0.089781299	-0.089781299	1.90×10^{-11}	3.70×10^{-10}
0.4	-0.1583616	-0.158361599	-0.158361599	5.00×10^{-10}	1.00×10^{-9}
0.5	-0.2421875	-0.242187496	-0.242187499	7.00×10^{-10}	3.10×10^{-9}
0.6	-0.3320064	-0.332006390	-0.332006399	8.00×10^{-10}	1.00×10^{-8}
0.7	-0.4076457	-0.407645676	-0.407645699	7.00×10^{-10}	2.30×10^{-8}
0.8	-0.4302848	-0.430284748	-0.430284799	2.00×10^{-10}	5.15×10^{-8}
0.9	-0.3317031	-0.331702995	-0.331703100	5.00×10^{-10}	1.05×10^{-10}
1	0	1.986517×10^{-7}	-3×10^{-10}	3.00×10^{-10}	1.99×10^{-7}

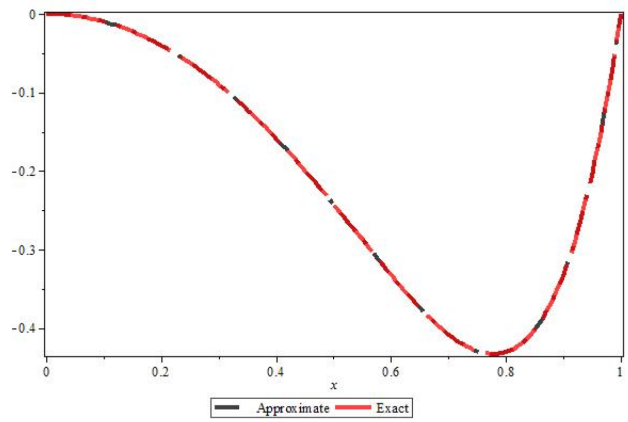

FIGURE 1. Bernoulli polynomials approximate solution

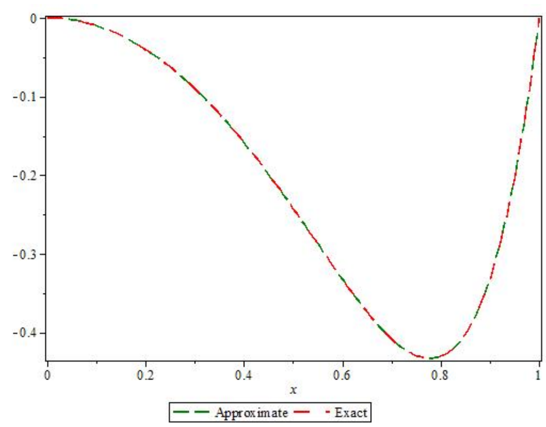

FIGURE 2. Shifted Legendre polynomials approximate solution

5.2: Approximate solution and error of example 2

x	exact	Shifted Legendre	Bernoulli Polynomial	Bernoulli Polynomial Error	Shifted Legendre Error
0	0	-9.75871×10^{-11}	-2.60251×10^{-10}	2.60×10^{-10}	9.75×10^{-11}
0.1	0.001	0.0010000118	0.0010000014	1.46×10^{-9}	1.18×10^{-8}
0.2	0.008	0.0080000496	0.0080000067	6.71×10^{-9}	4.96×10^{-8}
0.3	0.027	0.0270001318	0.0270000179	1.79×10^{-8}	1.31×10^{-7}
0.4	0.064	0.0640002905	0.0640000393	3.93×10^{-8}	2.90×10^{-7}
0.5	0.125	0.1250005797	0.1250000785	7.85×10^{-8}	5.79×10^{-7}
0.6	0.216	0.2160010937	0.2160001473	1.47×10^{-7}	1.09×10^{-6}
0.7	0.343	0.3430019937	0.3430002672	2.67×10^{-7}	1.99×10^{-6}
0.8	0.512	0.5120035524	0.5120004734	4.73×10^{-7}	3.55×10^{-6}
0.9	0.729	0.7290062324	0.7290008253	8.25×10^{-7}	6.23×10^{-6}
1	1.000	1.000010828	1.000001424	1.42×10^{-6}	1.08×10^{-5}

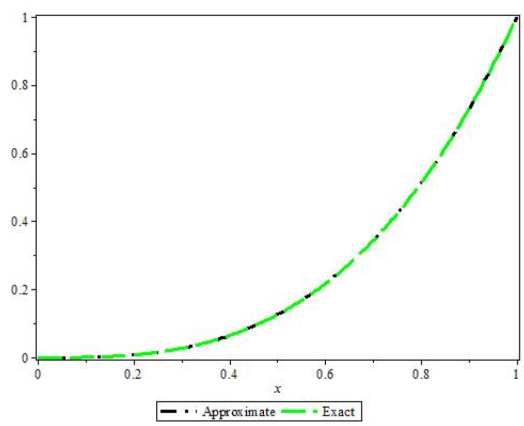

FIGURE 3. Bernoulli polynomials approximate solution

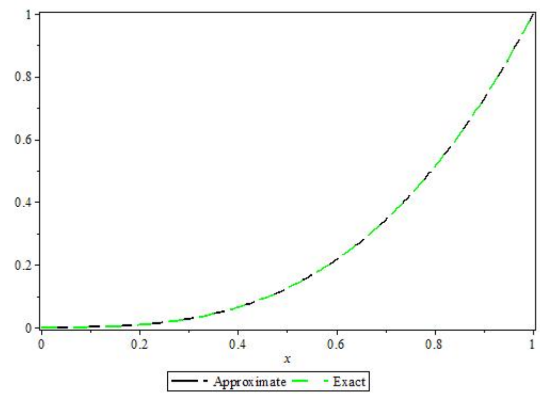

FIGURE 4. Bernoulli polynomials approximate solution

5.3: Approximate and error solution of example 3

x	exact	Shifted Legendre	Bernoulli Polynomial	Bernoulli Polynomial Error	Shifted Legendre Error
0	0	1.01300×10^{-10}	-1.13236×10^{-10}	1.13×10^{-10}	1.01300×10^{-10}
0.1	0.001	0.0009999998	0.0010000001	1.47×10^{-10}	2.00×10^{-10}
0.2	0.008	0.0079999987	0.0080000013	1.30×10^{-9}	1.27×10^{-9}
0.3	0.027	0.0269999901	0.0270000086	8.68×10^{-9}	9.85×10^{-9}
0.4	0.064	0.0639999470	0.0640000414	4.14×10^{-8}	5.30×10^{-8}
0.5	0.125	0.1249997964	0.1250001484	1.48×10^{-7}	2.03×10^{-7}
0.6	0.216	0.2159993809	0.2160004318	4.31×10^{-7}	6.19×10^{-7}
0.7	0.343	0.3429984061	0.3430010800	1.08×10^{-6}	1.59×10^{-6}
0.8	0.512	0.5119963718	0.5120024083	2.40×10^{-6}	3.62×10^{-6}
0.9	0.729	0.7289924860	0.7290049128	4.91×10^{-6}	7.51×10^{-6}
1	1.000	0.9999855603	1.000009334	9.33×10^{-6}	1.44×10^{-5}

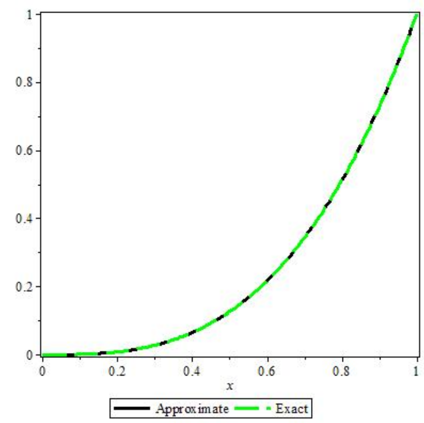

FIGURE 5. Bernoulli polynomials approximate solution

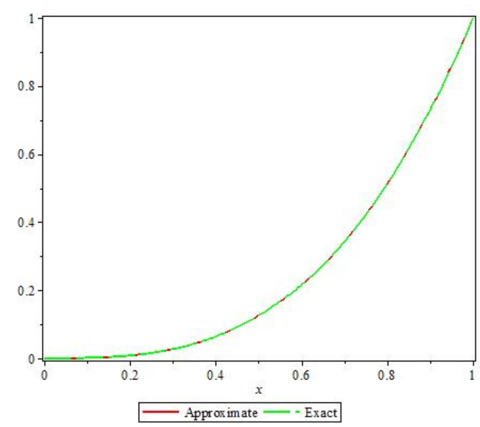

FIGURE 6. Bernoulli polynomials approximate solution

Conclusion. In this research work, three different examples of multi order fractional differential equation are solved using two basis function namely: Bernoulli polynomials and shifted Legendre polynomials. It was observed generally that approximate solutions obtained by using Bernoulli polynomial basis function were more close to the exact solutions than that of shifted Legendre polynomials. Though, the proposed method is suitable for solving multi order fractional differential equation for which they were applied with two different basis function. From the results obtained, it can be shown that Bernoulli polynomials gives a better approximation than shifted Legendre polynomials basis functions. As the case N is increasing the error obtained is getting more accurate.

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