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A Halpern Resolvent-Subgradient-Extragradients Method for finding common solution of fixed point, h-fixed point, equilibrium and variational inequality problems in Banach spaces

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Abstract

The key idea behind solving well known applicable problems is to reformulate the problems as finding a fixed point of a suitably defined mapping. The theory of variational inequalities and equilibrium problems involving some operators with monotonicity conditions have received remarkable development with several methods suggested for solving them. Efforts have significantly been made in approximating fixed points and h-fixed point of (possibly set valued, nonlinear) mapping in different suitable real Banach spaces. The intersections of these four streams; functional-analytic fixed and h- fixed point theories, and the mathematics of variational and equilibrium problems have not been well researched. The purpose of this paper is to construct an algorithm for the approximation of the common solution of the set of h-fixed points of a h-pseudo-contractive mapping; set of fixed point of a Bregman quasi mapping; set of solution of equilibrium problems for pseudomonotone mapping and the set of variational inequality problem for Lipschitz monotone mapping which leads to a prove of strong convergence theorem for it in Banach spaces.

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1. Introduction and Preliminaries

Many science, engineering problems are known to be formulated as fixed points, h-fixed points, variational inequalities and equilibrium problems (see, for example, [1]-[8]). Iterative methods have remained a powerful tool for finding solutions regarding these problems and a plethora of methods have been proposed, analyzed and suggested by several authors (see, for example, [2]-[8] and references therein). Thus, this work is concerned with combining classical Halpern, resolvent, subgradient-extragradient and extragradient methods known for solving fixed points a various mappings; h-fixed points of h-pseudocontractive mapping; equilibrium problem of pseudomonotone mapping and variational inequalities of monotone mapping respectively having been motivated by ([1]-[8] and references therein). This method does not seek to compute two projections onto a closed and convex subset of the ambient space which formally was the practice in solving variational inequalities as this approach affects efficiency and high cost of the method. Rather, the work intends to calculate projection onto a closed and convex subset of the ambient space for the first and projection onto the subgradient half-space for the second to achieve better result [2]. In addition, Bregman distance with respect to convex functions has become useful in aiding efficiency and computational flexibility when solving many problems modeled in the form listed above (see [10], [13], [14], [15] for more information). The introduction of Bregman distance and not norm yields significant results in the literature. Thus, the iterative methods is designed with respect to Bregman distance induced by a convex function and not with respect to norm. Furthermore, in order to obtain strong convergence results as against weak convergence, this paper utilizes the Halpern method so that the combined iterative method suggested converges strong to the common solution of the aforementioned problems.

2. Definition and Preliminaries

Let X(respectively X^*) stand as reflexive real Banach (respectively dual) spaces. Let $||.||$ stand as the induced norm. Let C stand for as a non-empty, closed and convex subset of X.

Definition 2.1. Let $T : C \to C$ stands for any mapping. Observe that x is the fixed point of any mapping T and the set of this point is denoted as follows:

$$Fix(T) := \{x \in K : x = Tx\}. \tag{1}$$

Definition 2.2. Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function and let $x \in int(dom\, h)$. For any $y \in X$, we define the right-hand derivative by

$$h^o(x, y) = \lim_{t \to 0^+} \frac{h(x + ty) - h(x)}{t}, \tag{2}$$

where $\mathbb{R}$ is the set of all real numbers, $dom\,h$ and $int(dom\,h)$ is the domain interior domain of a convex function respectively. The function h is said to be Gâteaux differentiable at a point x if the limit in (2) exists for any $y \in X$.

By this definition, $h^o(x,y) = \nabla h(x)$ which is the gradient of a convex function h. (see, for more information [9],[10], [11] for more information).

Definition 2.3. Let h be a convex and Gâteaux differentiable function at x, then the bifunction $d_h(.,.) : dom\ h \times int(dom\ h) \to \mathbb{R}^+$ defined by

$$d_h(y,x) = h(y) - h(x) - \langle \nabla h(x), y - x\rangle \tag{3}$$

is the Bregman distance induced by the convex function h (see [12] and the references contain therein).

Definition 2.4. A map $T : K \to int(dom\ h)$ with respect to a convex function $h : X \to (-\infty, +\infty]$ is called

(1) . Bregman relatively nonexpansive (shortly,(BRNE))(see [12]) if the following conditions holds

$$d_h(p,Tx) \leq d_h(p,x), \forall x \in K, \forall p \in Fix(T) = \hat{Fix}(T)$$

(2) . Bregman quasi nonexpansive (shortly,(BQNE))(see [5]) if the following conditions holds

$$d_h(p,Tx) \leq d_h(p,x), \forall x \in K, \forall p \in Fix(T)$$

Definition 2.5. A mapping $A : X \to X^*$ is monotone if for all $x, y \in dom(A)$, we have

$$\langle Ax - Ay, x - y\rangle \geq 0. \tag{4}$$

A mapping $A : X \to X^*$ is ρ-inverse strongly monotone if for all $x, y \in dom(A)$, we have

$$\langle Ax - Ay, x - y\rangle \geq \rho \|Ax - Ay\|^2. \tag{5}$$

Definition 2.6. For a given closed and convex subset $C \subset X$, a variational inequality problem for a mapping $A : X \to X^*$, is the problem of finding an element $u \in C$ such that

$$\langle Au, x - u\rangle \geq \forall x \in C. \tag{6}$$

The solution set of (6) will be denoted by $VI(C,A)$ (see [2] for more information).

Definition 2.7. $T : X \to X^*$ is said to be $h-$pseudo-contractive maps if for each $x, y \in X$, we have

$$\langle x - y, T(x) - T(y)\rangle \leq \langle x - y, \nabla h(x) - \nabla h(y)\rangle. \tag{7}$$

Also, T is called $\rho-$strictly $h-$pseudo-contractive maps if for each $x, y \in X$, we have

$$\langle x-y, T(x)-T(y)\rangle \leq \langle x-y, \nabla h(x)-\nabla h(y)\rangle - \rho\|(\nabla h(x)-\nabla h(y))-(T(x)-T(y))\|^2. \tag{8}$$

Observe that $h-$fixed point problem of this mapping is to find a fixed point say $q \in K$ such that,

$$Tq = \nabla h(q). \tag{9}$$

For more information regarding this mapping see [6].

Definition 2.8. Consider finding a point $z \in C$ such that mappings $g : C \times C \to R$ defined by

$$g(z, y) \geq 0, \forall y \in C. \tag{10}$$

This is called Equilibrium problem (EP) and the solution set of (10) will be denoted by EP(g) (see [4]-[5] for more information).

The bifunction $g : C \times C \to \mathbb{R}$ is said to satisfied $g(x, x) = 0, \forall x \in C$ and it is called monotone if $\forall x, u \in C$, we have $g(u, x) + g(x, u) \leq 0$. A bifuntion defined by $g(u, x) = \langle M(u), x - u\rangle$ is called pseudomonotone on C $if and only if$ M is pseudomonotone. In order to successfully solve arsing Equilibrium Problems (EP) with the involvement of pseudomonotone bifunction, the following conditions are needed:

- P1. $g(z, y) \geq 0 \implies g(y, z) \leq 0$;
- P2. g is Bregman-Lipschitz type continuous, that is, there exist two positive constants c_1, c_2 such that $g(x, y) + g(y, z) \geq g(x, z) - c_1 d_h(y, x) - c_2 d_h(z, y), \forall x, y, z \in C$;
- P3. g is weakly continuous on $C \times C$, that is, if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y respectively, then $g(x_n, y_n) \to g(x, y)$;
- P4. $g(x, .)$ is convex, lower-semicontinuous and sub-differential on C for every fixed $x \in C$;
- P5. for each $x, y, x \in C$, $\limsup_{t\downarrow} g(tx + (1-t)y, z) \leq g(y, z)$.

Definition 2.9. If h is strictly convex on $int(dom\ h)$ and C is a subset of X such that $int(dom\ h) \cap C \neq \emptyset$, then there exists at most one point $P_C^h(x) \in int(dom\ h) \cap C$ satisfying

$$d_h(P_C^h(x), x) = inf\left\{d_h(z, x) : z \in int(dom\ h) \cap C\right\}. \tag{11}$$

This point if exist is called the Bregman Projection of $x \in int(dom\ h)$ onto a nonempty closed and convex set C, (see [12] and the references therein).

Definition 2.10. Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous and convex function. The function $h^* : X^* \to \mathbb{R}$ defined by

$$h^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - h(x)\} \tag{12}$$

is called conjugate function of h. With this definition, we have that h^* is also proper, lower semi-continuous and convex, (see [9], [16], [11]).

Definition 2.11. A function $h : X \to (-\infty, +\infty]$ is said to be a Legendre function if it satisfies the following two conditions:

(L1) $int(dom\, h) \neq \emptyset$, h is Gâteaux differentiable on $int(dom\, h)$ and $dom\, f = int(dom\, h)$,

(L2) $int(dom h^*) \neq \emptyset$, h^* is Gâteaux differentiable on $int(dom h^*)$ and $dom h^* = int(dom\, h^*)$.

Remark. (a) Since X is reflexive, then we have that $(\partial h)^{-1} = \partial h^*$ and since h is Legendre, then ∂h is a bijection which satisfies $\nabla h = (\nabla h^*)^{-1}$, $ran \nabla h = dom \nabla h^* = int(dom h^*)$ and $ran \nabla h^* = dom \nabla h = int(dom h)$. h and h^* are Gâteaux differentiable and strictly convex on their $int(dom\, h)$ and $dom\, h$ respectively. If the subdifferential of h is single valued, it coincides with the gradient of h, that is $\partial h = \nabla h$.

(b) Example of a Legendre function is $h(x) = \frac{1}{p}\|x\|^p$, $(1 < p < \infty)$ and throughout this work, we assumed that h is Legendre, (see [17], [18] for more information).

Definition 2.12. The function h is called totally convex at $x \in int(dom\, h)$ if its modulus of total convexity at x, that is, the function $V_h : [0, \infty) \to [0, \infty)$ defined by

$$V_h(x, t) = \inf\{d_h(y, x) : y \in dom\, h, \|y - x\| = t\} \tag{13}$$

is non-negative whenever $t \geq 0$. The function h is called totally convex if it is totally convex at any point $x \in int(dom\, h)$. This property is less stringent than uniform convexity, (see [19], [20], [21] and references contain therein).

The function h is said to be totally convex on bounded sets ([19]) if $V_h(B, t) > 0$ for any nonempty bounded subset B of X and $t > 0$, where the modulus of total convexity of the function h on the set B is the function $V_h : int(dom\, h) \times [0, +\infty) \to [0, +\infty)$ defined by

$$V_h(B, t) := inf\, \{V_h(x, t) : x \in B \cap dom\, h\}\,. \tag{14}$$

Remark. It is easy to see that (3) is not symmetric and does not satisfy the well-known triangle inequality associated with classical distance metrics, but has the following nice properties (see [20], [14], [10]).

(B1) $d_h(.,x)$ for any $x \in int(dom\, h)$ is convex with respect to the first variable,
(B2) $d_h(x,x) = 0$,
(B3) $d_h(y,x)$ is positive if and only if $y \neq x$ for any $x \in int(dom\, h)$,
(B4) $d_h(x,y) + d_h(y,x) = \langle \nabla h(x), x - y\rangle - \langle \nabla h(y), x - z\rangle$ for any $x, y \in int(dom\, h)$,
(B5) $d_h(y,x) = d_h(y,z) + d_h(z,x) + \langle \nabla h(z), y - z\rangle - \langle \nabla h(x), y - z\rangle$ for any $x, y \in int(dom\, h)$ and for any $y \in dom\, h$.

3. Main results

In this section, we introduce and solve the problems $Fix(T) \cap VI(C,A) \cap EP(g) \cap Fix_h(S) \neq \emptyset$ and prove that sequence of iterates strongly converges to the feasible set of these problems. To achieve this, we need the following hypothesis and algorithm:

Conditions 3.1.

(H1) Let C represent a non-empty, closed and convex subset of a reflexive real Banach space X and its dual space X^*;
(H2) let $h : X \to (-\infty, +\infty]$ represent a strongly coercive Legendre function which is bounded, uniform Fréchet differentiable and totally convex on bounded subsets of X;
(H3) let $T : X \to X$, represent continuous Bregman quasi nonexpansive mapping such that $(I - T)$, is demi-closed at the origin;
(H4) let $S : X \to X^*$, represent continuous h-pseudo-contractive mapping;
(H5) let $g : d_n \times d_n \to \mathbb{R}$ be a bifunctionals satisfying P1 - P5; where d_n is a half-space, the bounding hyperplane, which supports the non-empty closed and convex subset C of X;
(H6) let $A : C \to X^*$ be a Lipschitz monotone mapping with Lipschitz constant L;
(H7) let $C \subset int(dom\, h)$, where $int(dom\, h)$ is the interior domain of the Legendre function;
(H8) let the common solution set be denoted by Ω be a non-empty set, that is $\Omega := Fix(T) \cap VI(C,A) \cap EP(g) \cap Fix_h(S) \neq \emptyset$.

Algorithm 3.1.
Initialize $u, x_0 \in X$. Let the sequence $\{x_n\}$ be generated by the following algorithm:

$$
(15) \quad \begin{cases}
z_n = \nabla h^* \left(a_n \nabla h\left(x_n\right) + (1-a_n)\nabla h\left(Tx_n\right)\right), \\
y_n = P^h_C \nabla h^*(\nabla h(z_n) - \lambda_n A z_n), \\
d_n = \left\{z \in X : \langle z - y_n, \nabla h(z_n) - \lambda_n A z_n - \nabla h(y_n)\rangle \leq 0\right\}, \\
e_n = P^h_{d_n} \nabla h^*(\nabla h(z_n) - \lambda_n A y_n), \\
u_n = argmin_{a \in d_n} \left\{\mu_n g(e_n, a) + d_h(a, e_n)\right\}, \\
w_n = argmin_{a \in d_n} \left\{\mu_n g(u_n, a) + d_h(a, e_n)\right\}, \\
v_n = R^{h, r_n}_S x_n, \\
x_{n+1} = \nabla h^* \left(\alpha_n \nabla h(u) + \beta_n \nabla h(x_n) + \gamma_n \nabla h(w_n) + \zeta_n \nabla h(v_n)\right), n \geq 0,
\end{cases}
$$

where ∇h is the gradient of h on X and $(r_n) \subset [c, \infty)$, $\{a_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\zeta_n\} \in (0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \zeta_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and the Lipschitz constant L.

Lemma 3.1. . *let h be a strongly convex function with constant $mu > 0$. Then for all $e_n \in dom(h)$ and $z_n \in intdom(h)$, the inequality holds:*

$$
(16) \quad d_h(e_n, z_n) \geq \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + \langle e_n - y_n, \nabla h(y_n) - \nabla h(z_n)\rangle.
$$

Proof. Invoking the identities of Bregman distance with respect to h given in Remark (2) and a Lemma in [22], we obtain

$$
\begin{aligned}
d_h(e_n, z_n) &= d_h(e_n, y_n) + d_h(y_n, z_n) + \langle e_n - y_n, \nabla h(y_n) - \nabla h(z_n)\rangle \\
&\geq \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + \langle e_n - y_n, \nabla h(y_n) - \nabla h(z_n)\rangle.
\end{aligned}
$$

□

Lemma 3.2. . *Given that **Conditions 3.1** holds true, then the sequences and that generated by the algorithm (15) are bounded.*

Proof. Let $\Upsilon_n = \nabla h^*(\nabla h(z_n) - \lambda_n A y_n)$. So $e_n = P^h_{d_n}\Upsilon_n$. Let $\ell \in \Omega$ and invoking a Lemma in [12] we obtain

$$
\begin{aligned}
d_h(\ell, e_n) &= d_h(\ell, \Upsilon_n) - d_h(e_n, \Upsilon_n)\\
&= h(\ell) - h(\Upsilon_n) - \langle \nabla h(\Upsilon_n), \ell - \Upsilon_n\rangle - [h(e_n) - h(\Upsilon_n) - \langle \nabla h(\Upsilon_n), e_n - \Upsilon_n\rangle)]\\
&= h(\ell) - \langle \nabla h(\Upsilon_n), \ell - e_n\rangle - h(e_n)\\
&= h(\ell) - \langle \nabla h(z_n) - \lambda_n A y_n, \ell - e_n\rangle - h(e_n)\\
&= h(\ell) - \langle \nabla h(z_n), \ell - e_n\rangle + \langle \lambda_n A y_n, \ell - e_n\rangle - h(e_n)\\
&= h(\ell) - \langle \nabla h(z_n), \ell - z_n\rangle + \langle \nabla h(z_n), e_n - z_n\rangle + \langle \lambda_n A y_n, \ell - e_n\rangle - h(e_n)\\
&= d_h(\ell, z_n) - d_h(e_n, z_n) + \langle \lambda_n A y_n, \ell - e_n\rangle\\
&= d_h(\ell, z_n) - d_h(e_n, z_n) + \langle \lambda_n A y_n, \ell - y_n\rangle + \langle \lambda_n A y_n, y_n - e_n\rangle\\
&= d_h(\ell, z_n) - d_h(e_n, z_n) + \lambda_n\langle A y_n - A\ell, \ell - y_n\rangle + \lambda_n\langle A\ell, \ell - y_n\rangle + \langle \lambda_n A y_n, y_n - e_n\rangle\\
&\leq d_h(\ell, z_n) - d_h(e_n, z_n) + \langle \lambda_n A y_n, y_n - e_n\rangle\\
&\leq d_h(\ell, z_n) - \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + \langle e_n - y_n, \nabla h(y_n) - \nabla h(z_n)\rangle + \langle \lambda_n A y_n, y_n - e_n\rangle\\
&\leq d_h(\ell, z_n) - \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + \langle y_n - e_n, \lambda_n A y_n + \nabla h(y_n) - \nabla h(z_n)\rangle\\
&= d_h(\ell, z_n) - \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + \langle e_n - y_n, \lambda_n A z_n - \lambda_n A y_n\rangle\\
&+ \langle e_n - y_n, \nabla h(z_n) - \lambda_n A z_n - \nabla h(y_n)\rangle\\
&\leq d_h(\ell, z_n) - \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + \lambda_n\|e_n - y_n\|\|A z_n - A y_n\|\\
&\leq d_h(\ell, z_n) - \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + L.\lambda_n\|e_n - y_n\|\|z_n - y_n\|\\
&\leq d_h(\ell, z_n) - \frac{\mu}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] + \frac{1}{2}L.\lambda_n\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right]
\end{aligned}
$$

(17)

$$
= d_h(\ell, z_n) - \frac{(\mu - L.\lambda_n)}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right].
$$

But from ([10]), we obtain

$$
\begin{aligned}
d_h(\ell, z_n) &= d_h(\ell, \nabla h^*\left(a_n \nabla h\left(x_n\right) + (1 - a_n)\nabla h\left(T x_n\right)\right))\\
&\leq d_h(\ell, x_n) - a_n(1 - a_n)\rho^*\left(\|\nabla h(x_n) - \nabla h(T x_n)\|\right).
\end{aligned} \tag{18}
$$

Substituting (18) into (17), we have that

(19)

$$
d_h(\ell, e_n) \leq d_h(\ell, x_n) - a_n(1 - a_n)\rho^*\left(\|\nabla h(x_n) - \nabla h(T x_n)\|\right) - \frac{(\mu - L.\lambda_n)}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right].
$$

Now using a Lemma in [4], [22] and substituting 19, we obtain

$$
\begin{aligned}
d_h(\ell, w_n) &\leq d_h(\ell, e_n) - (1-\mu_n c_1) d_h(u_n, e_n) - (1-\mu_n c_2) d_h(w_n, u_n) \\
&\leq d_h d_h(\ell, x_n) - a_n(1-a_n)\rho^* \left(\|\nabla h(x_n) - \nabla h(Tx_n)\|\right) - \frac{(\mu - L.\lambda_n)}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] \\
&- (1-\mu_n c_1)\frac{\mu}{2}\|u_n - e_n\|^2 - (1-\mu_n c_2)\frac{\mu}{2}\|w_n - u_n\|^2 \\
&\leq d_h(\ell, x_n) - a_n(1-a_n)\rho^* \left(\|\nabla h(x_n) - \nabla h(Tx_n)\|\right) - \frac{(\mu - L.\lambda_n)}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] \\
&- \frac{\mu}{2}\left((1-\mu_n c_1)\|u_n - e_n\|^2 + (1-\mu_n c_2)\right)\|w_n - u_n\|^2.
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
d_h(\ell, v_n) &\leq d_h(\ell, x_n) - d_h(v_n, x_n) \\
&\leq d_h(\ell, x_n) - \frac{\mu}{2}\|v_n - x_n\|^2.
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
d_h(\ell, x_{n+1}) &= d_h(\ell, \nabla h^* \left(\alpha_n \nabla h(u) + \beta_n \nabla h(x_n) + \gamma_n \nabla h(w_n) + \zeta_n \nabla h(v_n)\right)) \\
&\leq \alpha_n d_h(\ell, u) + \beta_n d_h(\ell, x_n) + \gamma_n d_h(\ell, e_n) + \zeta_n d_h(\ell, v_n) \\
&\leq \alpha_n d_h(\ell, u) + \beta_n d_h(\ell, x_n) + \gamma_n d_h(\ell, x_n) - \gamma_n a_n(1-a_n)\rho^* \left(\|\nabla h(x_n) - \nabla h(Tx_n)\|\right) \\
&- \gamma_n \frac{(\mu - L.\lambda_n)}{2}\left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2\right] \\
&- \zeta_n \frac{\mu}{2}\left((1-\mu_n c_1)\|u_n - e_n\|^2 + (1-\mu_n c_2)\|w_n - u_n\|^2 + \|v_n - x_n\|^2\right) \\
&\leq \alpha_n d_h(\ell, u) + (1-\beta_n) d_h(\ell, x_n) \\
&\leq \max\left\{d_h(\ell, u), d_h(\ell, x_n)\right\} \\
&\ \vdots \\
&\leq \max\left\{d_h(\ell, u), d_h(\ell, x_0)\right\} \text{ for all } n \geq 0.
\end{aligned} \tag{22}
$$

This means by implication that $\{d_h(\ell, x_n)\}$ is bounded. Following from ([20]) $\{x_n\}$ is bounded and consequently $\{z_n\}, \{y_n\}, \{d_n\}, \{e_n\}, \{u_n\}, \{w_n\}, \{v_n\}$ are all bounded. ☐

Theorem 3.3. *Given that **Conditions 3.1** holds true, then the sequences and that generated by the algorithm (15) converges strongly to a common point of Ω which is a Projection of u onto Ω with respect to Bregman distance h.*

Proof. Let $\ell = P_{\Omega}^{h}u$ be the Projection of u onto Ω with respect to Bregman distance h. Now invoking (17), (20), (21) and a Lemma in [6] we obtain

$$\begin{aligned}
d_h(\ell, x_{n+1}) &= d_h(\ell, \nabla h^* (\alpha_n \nabla h(u) + \beta_n \nabla h(x_n) + \gamma_n \nabla h(w_n) + \zeta_n \nabla h(v_n))) \\
&= v_h(\ell, \alpha_n \nabla h(u) + \beta_n \nabla h(x_n) + \gamma_n \nabla h(w_n) + \zeta_n \nabla h(v_n)) \\
&= h(\ell) - \langle \ell, \alpha_n \nabla h(u) + \beta_n \nabla h(x_n) + \gamma_n \nabla h(w_n) + \zeta_n \nabla h(v_n) \rangle \\
&+ h^*(\alpha_n \nabla h(u) + \beta_n \nabla h(x_n) + \gamma_n \nabla h(w_n) + \zeta_n \nabla h(v_n)) \\
&\leq \alpha_n h(\ell) + \beta_n \langle \ell, \nabla h(x_n) \rangle + \gamma_n \langle \ell, \nabla h(w_n) \rangle + \beta_n \langle \ell, \nabla h(v_n) \rangle \\
&+ \beta_n h^*(\nabla h(x_n)) + \beta_n h^*(\nabla h(w_n)) + \zeta_n h^*(\nabla h(v_n)) - \alpha_n \langle x_{n+1} - \ell, \nabla h(\ell) - \nabla h(u) \rangle \\
&= \alpha_n V_h(\ell, \ell) + \beta_n V_h(\ell, \nabla h(x_n)) + \gamma_n V_h(\ell, \nabla h(w_n)) + \zeta_n V_h(\ell, \nabla h(v_n)) \\
&- \alpha_n \langle x_{n+1} - \ell, \nabla h(\ell) - \nabla h(u) \rangle \\
&= \alpha_n d_h(\ell, \ell) + \beta_n d_h(\ell, x_n) + \gamma_n d_h(\ell, w_n) + \zeta_n d_h(\ell, v_n) - \alpha_n \langle x_{n+1} - \ell, \nabla h(\ell) - \nabla h(u) \rangle \\
&= (1 - \alpha_n) d_h(\ell, x_n) - \gamma_n a_n (1 - a_n) \rho^* (\|\nabla h(x_n) - \nabla h(T x_n)\|) \\
&- \gamma_n \frac{(\mu - L.\lambda_n)}{2} \left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2 \right] \\
&- \zeta_n \frac{\mu}{2} \left((1 - \mu_n c_1) \|u_n - e_n\|^2 + (1 - \mu_n c_2) \|w_n - u_n\|^2 + \|v_n - x_n\|^2 \right) \\
&+ \alpha_n \langle x_{n+1} - \ell, \nabla h(u) - \nabla h(\ell) \rangle \qquad (23) \\
&\leq (1 - \alpha_n) d_h(\ell, x_n) + \alpha_n \langle x_n - \ell, \nabla h(u) - \nabla h(\ell) \rangle + \alpha_n \langle x_{n+1} - x_n, \nabla h(\ell) - \nabla h(u) \rangle \qquad (24) \\
&\leq (1 - \alpha_n) d_h(\ell, x_n) + \alpha_n \langle x_n - \ell, \nabla h(u) - \nabla h(\ell) \rangle + \alpha_n \|x_{n+1} - x_n\| \|\nabla h(\ell) - \nabla h(u)\|.
\end{aligned}$$

Now, establish strong convergence of the sequence to a common point of Ω, we examine two cases.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that $\{d_h(\ell, x_n)\}$ is decreasing for all $n \geq n_0$. It then follows that is convergent $\{d_h(\ell, x_n)\}$ and hence $d_h(\ell, x_n) - d_h(\ell, x_{n+1}) \to 0$ as $n \to \infty$. Thus, from (23) and applying the conditions on $\{a_n\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\zeta_n\}$ of (15) we obtain that

$$\lim_{n \to \infty} \left[\|e_n - y_n\|^2 + \|z_n - y_n\|^2 \right] = 0, \tag{25}$$

$$\lim_{n \to \infty} \left(\|u_n - e_n\|^2 + \|w_n - u_n\|^2 + \|v_n - x_n\|^2 \right) = 0, \tag{26}$$

and,

$$\lim_{n \to \infty} \|\nabla h(x_n) - \nabla h(T x_n)\| = 0. \tag{27}$$

This means by implication that from (25) and (26),

$$\lim_{n \to \infty} \|e_n - y_n\| = \lim_{n \to \infty} \|z_n - y_n\| = 0 \text{ and thus } \lim_{n \to \infty} \|z_n - e_n\| = 0, \tag{28}$$

$$\lim_{n\to\infty}\|u_n-e_n\|=\|w_n-u_n\|=\|v_n-w_n\|=0 \text{ and thus } \lim_{n\to\infty}\|x_n-w_n\|=0, \lim_{n\to\infty}\|x_n-u_n\|=0. \tag{29}$$

Since from [10], ∇h is uniformly norm-to-norm continuous on bounded subset of X^*, we obtain from (27) that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0 \text{ implies } \lim_{n\to\infty}d_h(x_n,Tx_n)=0. \tag{30}$$

Now from (15), $z_n=\nabla h^*\left(a_n\nabla h\left(x_n\right)+\left(1-a_n\right)\nabla h\left(Tx_n\right)\right)$, so that

$$\|z_n-x_n\|=(1-a_n)\|Tx_n-x_n\| \tag{31}$$

Applying the condition on α_n and invoking (30), we obtain that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{32}$$

Combining (32) and (28), we obtain that

$$\lim_{n\to\infty}\|x_n-e_n\|=0. \tag{33}$$

Now,

$$\begin{aligned}\|\nabla h(x_{n+1})-\nabla h(x_n)\|&=\|\alpha_n(\nabla h(u)+\beta_n\nabla h(x_n)+\gamma_n\nabla h(e_n)+\zeta_n\nabla h(v_n)-\nabla h(x_n))\|\\&\le\alpha_n\|\nabla h(u)-\nabla h(x_n)\|+\gamma_n\|\nabla h(e_n)-\nabla h(x_n)\|+\zeta_n\|\nabla h(v_n)-\nabla h(x_n)\|.\end{aligned}$$

Invoking (29), (33), condition on α_n and the uniformity of ∇h in [10], we obtain that

$$\lim_{n\to\infty}\|\nabla h(x_{n+1})-\nabla h(x_n)\|=0. \tag{34}$$

In addition, by the uniformity of ∇h^* (see [10]), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{35}$$

Since $\{x_n\}$ is a bounded sequence of X, we can find a point l and a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that x_{n_j} converge weakly to l. Invoking (28), (29), (32), (33), we obtain that the sequences z_n,e_n,w_n,u_n,v_n converge weakly to l. Hence by the continuity property of T, we have from (30) that $l\in Fix(T)$. Next, since $\{x_n\}$ is a bounded sequence of X and $y_n=P_C^h\nabla h^*(\nabla h(z_n)-\lambda_nAz_n)$

$$\begin{aligned}\langle\lambda_nAy_n,y_n-\ell\rangle&=\langle\ell-y_n,\nabla h(y_n)-\nabla h(z_n)\rangle+\langle\lambda_nAy_n,y_n-\ell\rangle\\&\le\langle y_n-\ell,\lambda_nAy_n+\nabla h(y_n)-\nabla h(z_n)\rangle\\&=\langle\ell-y_n,\lambda_nAz_n-\lambda_nAy_n\rangle+\langle\ell-y_n,\nabla h(z_n)-\lambda_nAz_n-\nabla h(y_n)\rangle\\&\le\langle\ell-y_n,\lambda_nAz_n-\lambda_nAy_n\rangle\\&\le\|\ell-y_n\|\lambda_n\|Az_n-Ay_n\|\\&\le\|\ell-y_n\|\lambda_n\|z_n-y_n\|.\end{aligned} \tag{36}$$

It follows that $\limsup_{n\to\infty}\langle \lambda_n A y_n, y_n - \ell\rangle \leq 0$. Invoking the monotonicity of A, we have that

$$\begin{aligned} \langle \lambda_n A\ell, \ell - l\rangle &= \limsup_{n\to\infty}\langle \lambda_n A\ell, y_n - \ell\rangle \\ &\leq \limsup_{n\to\infty}\langle \lambda_n A y_n, y_n - \ell\rangle \leq 0. \end{aligned} \tag{37}$$

From the foregoing analysis, we obtain that $l \in VI(C, A)$. Next, we show that $l \in EP(g)$. Since

$$u_n = argmin_{a\in d_n}\left\{\mu_n g(e_n, a) + d_h(a, e_n)\right\},$$

then invoking [5], [6] and condition P4, we obtain that $0 \in \delta\mu_n g(e_n, u_n) + \nabla d_h(u_n, e_n) + N_{d_n}(u_n)$. So there exist $\theta_n \in \delta g(e_n, u_n)$ and $\Theta_n \in N_{d_n}(u_n)$ such that

$$\mu_n\theta_n + \nabla h(u_n) - \nabla h(e_n) + \Theta_n = 0. \tag{38}$$

Observe that $\Theta_n \in N_{d_n}(u_n)$ and $\langle q - u_n, \Theta_n\rangle \leq 0$ for all $q \in d_n$. Since $\theta_n \delta g(e_n, u_n)$, we have

$$g(e_n, q) - g(e_n, u_n) \geq \langle q - u_n, \Theta_n\rangle, \tag{39}$$

for all $q \in d_n$. Invoking (38) and (39), we have that

$$[g(e_n, q) - g(e_n, u_n)] \geq \frac{1}{\mu_n}\langle u_n - q, \nabla h(u_n) - \nabla h(e_n), \tag{40}$$

for all $q \in d_n$. Now by the fact that g is weakly continuous using $L3$ condition and combining with (29) and the condition on μ_n, we obtain that

$$g(l, q) \geq 0, \tag{41}$$

for all $q \in d_n$. Hence $l \in EP(g)$.

Furthermore, we show that $l \in Fix_h(S)$. Let $v_{n_k} = R_S^{h,r_{n_k}}(x_n)$. Invoking [6], we obtain

$$\langle y - v_{n_k}, Sv_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - v_{n_k}, (1 + r_{n_k})\nabla h(v_{n_k}) - \nabla h(x_{n_k})\rangle \leq 0.$$

By the convexity of C, we set $y_\alpha = \alpha y + (1-\alpha)q \in C$, for $y \in C, \alpha \in [0, 1]$. Thus, applying similar approach in [8], we arrive at

$$\langle l - y_\alpha, Sy_\alpha\rangle \geq \langle l - y_\alpha, \nabla h(y_\alpha)\rangle.$$

Hence, $l \in Fix_h(S)$.

Next, we prove that $\{x_n\}$ converges strongly to a common point of Ω which is a Projection of u onto Ω, that is $\ell = P_\Omega^h u$. Now since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to l and

$$\lim_{n\to\infty}\langle x_{n+1} - \ell, \nabla h(u) - \nabla h(\ell)\rangle = \lim_{j\to\infty}\langle x_{n_j+1} - \ell, \nabla h(u) - \nabla h(\ell)\rangle. \tag{42}$$

Since $l \in \Omega$, we obtain from [12] and (42) that

$$\lim_{n\to\infty} \langle x_{n+1} - \ell, \nabla h(u) - \nabla h(\ell)\rangle = c\langle l - \ell, \nabla h(u) - \nabla h(\ell)\rangle \leq 0. \tag{43}$$

From (24), we have that

$$d_h(\ell, x_{n+1}) \leq (1-\alpha_n) d_h(\ell, x_n) + \alpha_n \langle x_n - \ell, \nabla h(u) - \nabla h(\ell)\rangle + \alpha_n \|x_{n+1} - x_n\| \|\nabla h(\ell) - \nabla h(u)\| \tag{44}$$

By applying a Lemma in [23] on (44), we conclude that $d_h(\ell, x_n) \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $\ell = P^h_\Omega u$.

Case 2. Suppose that $\{d_h(\ell, x_n)\}$ is not a monotone decreasing sequence (see [24]). We let $\phi : \mathbb{N} \to \mathbb{N}$ be a mapping defined for $n \geq n_0$ for some sufficiently large n_0 by

$$\phi(n) = \max\left\{i \in \mathbb{N} : i \leq n, d_h(\ell, x_{n_j}) \leq d_h(\ell, x_{n_i+1})\right\}. \tag{45}$$

Then ϕ is a non-decreasing sequence such that $\phi(n) \to \infty$ and $d_h(\ell, x_{\phi(n)}) \leq d_h(\ell, x_{\phi(n)+1})$ is convergent for $n \geq n_0$. This implies that

$$\|x_{\phi(n)} - x_{\phi(n)+1}\| \to 0 \text{ as } \phi(n) \to \infty. \tag{46}$$

Following the same line of argument as in the **Case 1**,

$$\lim_{\phi(n)\to\infty} \langle x_{\phi(n)+1} - \ell, \nabla h(u) - \nabla h(\ell)\rangle = \langle l - \ell, \nabla h(u) - \nabla h(\ell)\rangle \leq 0. \tag{47}$$

From (24), we have that

$$\begin{aligned} d_h(\ell, x_{\phi(n)+1}) &\leq (1-\alpha_{\phi(n)}) d_h(\ell, x_{\phi(n)}) + \alpha_{\phi(n)} \langle x_{\phi(n)} - \ell, \nabla h(u) - \nabla h(\ell)\rangle \\ &\quad + \alpha_{\phi(n)} \|x_{\phi(n)+1} - x_{\phi(n)}\| \|\nabla h(\ell) - \nabla h(u)\|. \end{aligned} \tag{48}$$

Since $\alpha_{\phi(n)} > 0$, we obtain from (48), (47) and (46) that

$$\begin{aligned} d_h(\ell, x_{\phi(n)}) &\leq \alpha_{\phi(n)} \langle x_{\phi(n)} - \ell, \nabla h(u) - \nabla h(\ell)\rangle \\ &\leq 0 \end{aligned} \tag{49}$$

By applying a Lemma [23] on (49), we conclude that $d_h(\ell, x_{\phi(n)}) \to 0$ as $\phi(n) \to \infty$. This suggest that $\left\{x_{\phi(n)}\right\}$ converges ℓ. Therefore, combining **Case 1** and **Case 2**, we conclude that $\{x_n\}$ converges strongly to $\ell = P^h_\Omega u$. □

Remark. In **Algorithm 3.1**, if we assume the following conditions

(H1) let $T : C \to C$, represent continuous Bregman quasi nonexpansive mapping such that $(I - T)$, is demi-closed at the origin;
(H2) let $S : C \to X^*$, represent continuous h-pseudo-contractive mapping;
(H3) let $g : C \times C \to \mathbb{R}$ be a bifunctionals satisfying P1 - P5; where C is a non-empty closed and convex subset X;

then we have
Initialize $u, x_0 \in C$. Let the sequence $\{x_n\}$ be generated by the following algorithm:

$$(50)\quad \begin{cases} z_n = \nabla h^* \left(a_n \nabla h\left(x_n\right) + \left(1-a_n\right) \nabla h\left(Tx_n\right)\right), \\ y_n = P_C^h \nabla h^*(\nabla h(z_n) - \lambda_n A z_n), \\ d_n = \left\{z \in X : \langle z - y_n, \nabla h(z_n) - \lambda_n A z_n - \nabla h(y_n)\rangle \leq 0\right\}, \\ e_n = P_{d_n}^h \nabla h^*(\nabla h(z_n) - \lambda_n A y_n), \\ u_n = argmin_{a \in C}\left\{\mu_n g(x_n, a) + d_h(a, x_n)\right\}, \\ w_n = argmin_{a \in C}\left\{\mu_n g(u_n, a) + d_h(a, x_n)\right\}, \\ v_n = R_S^{h, r_n} w_n, \\ x_{n+1} = \nabla h^*\left(\alpha_n \nabla h(u) + \beta_n \nabla h(x_n) + \gamma_n \nabla h(e_n) + \zeta_n \nabla h(v_n)\right), n \geq 0, \end{cases}$$

Conclusion. In this paper, algorithm for the approximation of the common solution of the set of h-fixed points of a h-pseudo-contractive mapping; set of fixed point of a Bregman quasi mapping; set of solution of equilibrium problems for pseudomonotone mapping and the set of variational inequality problem for Lipschitz monotone mapping was constructed and analyzed. A strong convergence theorem for it was proved specifically in reflexive Banach spaces. The constructed algorithm and the result obtained was found to improve the result of [2]-[3], [7] from Hilbert space to reflexive Banach space and from solving one or two common problems to solving four common problems of various set valued, nonlinear mappings. Additionally, Theorem 3.3 complements Theorem **3.2** of [5] and [6] but improved their results from solving three common problems to solving four common problems of set valued, nonlinear mappings.

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