



A family of harmonic univalent functions defined by a linear multiplier fractional operator

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ABSTRACT

In this paper, we use a generalized linear multiplier fractional differential operator to define a new family of harmonic univalent functions in the unit disk. Also, we obtain the coefficient conditions, convolution condition and convex combination for the class.

1. INTRODUCTION

Let A_μ denote the class of functions of the form:

$$(1) \quad f(z) = z^{\mu+1} + \sum_{k=2}^{\infty} a_k z^{\mu+k}, \quad 0 \leq \mu < 1$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

If for convenience, we set $A_0 = A$, we see that A_0 is the usual class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

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which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. Moreover, we have $f(z) \in A \implies z^\mu f(z) \in A_\mu$, $0 \leq \mu < 1$.

Now expanding (1) by Binomial theorem, we define

$$(2) \quad f(z)^\rho = z^{\rho(\mu+1)} + \sum_{k=2}^{\infty} \rho a_k z^{(\rho-1)(\mu+1)+(\mu+k)}, \quad \rho > 0, \quad 0 \leq \mu < 1$$

2. PRELIMINARIES

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D . In any simply connected domain, we can write

$$(3) \quad f = h + \bar{g}$$

The form (3) is called the local decomposition of the harmonic function f , where h and g are analytic in D . We call h and g the analytic and coanalytic part of f respectively. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'| > |g'|$ in D .

Denote by S_U , the class of functions f of the form (3) that are harmonic univalent and sense-preserving in the unit disk U . The subclasses of harmonic univalent functions have been studied by some authors for different purposes and different properties, (See Juma *et al.* (2015), Ezhilarasi *et al.* (2014), Al-shaqsi *et al.* (2010), Fadipe-Joseph and Salami (2017), Serkan *et al.* (2018), Sharma and Ravindar (2018) and Shuhai Li *et al.* (2019)). Following the linear multiplier fractional differential operator $D_\lambda^{m,\alpha}$ for $f(z) \in A$ defined by Al-Oboudi and Al-Omoudi (2009), Noor *et al.* (2009) extends this definition to the harmonic function $f(z) = h(z) + \bar{g(z)}$ as follows:

$$(4) \quad \tilde{D}_\lambda^{m,\alpha} f(z) = \sum_{k=1}^{\infty} \phi_{k,m}(\alpha, \lambda)(a_k z^k + \overline{b_k z^k}), \quad 0 \leq \alpha < 1, \quad m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

where

$$\phi_{k,m}(\alpha, \lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1 + \lambda(k-1)) \right]^m$$

and

$$(5) \quad \left. \begin{aligned} h(z) &= z + \sum_{k=2}^{\infty} a_k z^k \\ g(z) &= \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1 \end{aligned} \right\}$$

We note that if $g \equiv 0$ then (4) becomes the linear multiplier fractional differential operator $D_\lambda^{m,\alpha}$, Al-Oboudi and Al-Omoudi (2009).

In this work, we express the analytic functions h and g as:

$$(6) \quad \left. \begin{aligned} h(z)^\rho &= z^{\rho(\mu+1)} + \sum_{k=2}^{\infty} \rho a_k z^{(\rho-1)(\mu+1)+(\mu+k)} \\ g(z)^\rho &= \sum_{k=1}^{\infty} \rho b_k z^{(\rho-1)(\mu+1)+(\mu+k)}, \quad |b_1| < 1 \end{aligned} \right\}$$

So that,

$$f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$$

where

$$(7) \quad \left. \begin{aligned} h(z) &= z^{(\mu+1)} + \sum_{k=2}^{\infty} a_k z^{(\mu+k)} \\ g(z) &= \sum_{k=1}^{\infty} b_k z^{(\mu+k)}, \quad |b_1| < 1 \text{ in the class } A_\mu \end{aligned} \right\}$$

and (7) reduces to (5) when $\mu = 0$.

In this present paper, we define and study a new class $\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)$, ($\rho > 0, 0 \leq \mu < 1, 0 \leq \alpha < 1, \zeta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0$) of harmonic univalent functions in $U = \{z \in \mathbb{C} : |z| < 1\}$ and a subclass of it.

3. THE CLASS $\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)$

Definition 3.1. Let $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$ be a harmonic function, where $h(z)^\rho$ and $g(z)^\rho$ are given by (6). Then $f(z)^\rho \in \Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)$ if it satisfies

$$(8) \quad \operatorname{Re} \left\{ \zeta \tilde{D}_\lambda^{m,\alpha} \left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}} \right)} \right] + \tilde{D}_\lambda^{m,\alpha} [h'(z)^\rho + \overline{g'(z)^\rho}] - \zeta \right\} > \gamma$$

$\rho > 0, 0 \leq \mu < 1, 0 \leq \alpha < 1, \zeta \geq 0, 0 \leq \gamma < 1, \lambda \geq 0, z \in U$, where

$$\tilde{D}_\lambda^{m,\alpha} f(z)^\rho = \sum_{k=1}^{\infty} \phi_{k,m}(\alpha, \lambda) (a_k z^{(\rho-1)(\mu+1)+(\mu+k)} + \overline{b_k z^{(\rho-1)(\mu+1)+(\mu+k)}})$$

and

$$\phi_{k,m}(\alpha, \lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1 + \lambda(k-1)) \right]^m, \quad m \in \mathbb{N}_0$$

Remark. We note that the class $\Psi_\gamma^{1,0}(m, \alpha, \zeta, \lambda) \equiv RH_\gamma(m, \alpha, \beta, \lambda)$ introduced and studied by Ezhilarasi *et al.* (2014).

Definition 3.2. We also let $\overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$ be the subclass of $\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)$ such that $h(z)^\rho$ and $g(z)^\rho$ are given by

$$(9) \quad \left. \begin{aligned} h(z)^\rho &= z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho |a_k| z^{(\rho-1)(\mu+1)+(\mu+k)} \\ g(z)^\rho &= - \sum_{k=1}^{\infty} \rho |b_k| z^{(\rho-1)(\mu+1)+(\mu+k)}, \quad |b_1| < 1 \end{aligned} \right\}$$

Theorem 3.3. Let $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$, where $h(z)^\rho, g(z)^\rho$ are given by (6). Furthermore, let

$$(10) \quad \sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) (|a_k| + |b_k|) \leq (\rho+1)(\mu+1) + \zeta - \frac{\gamma}{\rho}$$

$|b_1| < \frac{\rho(\mu+1)-\gamma}{\rho[\rho(\mu+1)+\zeta]} < 1$, where $a_1 = 1$, $\rho > 0$, $\zeta \geq 0$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$. Then $f(z)^\rho$ is harmonic univalent and sense preserving in U and $f(z)^\rho \in \Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)$.

Proof. For $z_1 \neq z_2$ and $|z_1| < |z_2| < 1$, we have by using (10),

$$\begin{aligned} |f(z_1)^\rho - f(z_2)^\rho| &\geq \left| z_1^{\rho(\mu+1)} - z_2^{\rho(\mu+1)} + \sum_{k=2}^{\infty} \rho \left\{ z_1^{(\rho-1)(\mu+1)+(\mu+k)} - z_2^{(\rho-1)(\mu+1)+(\mu+k)} \right\} a_k \right| \\ &\quad - \left| \sum_{k=1}^{\infty} \rho \left\{ z_1^{(\rho-1)(\mu+1)+(\mu+k)} - z_2^{(\rho-1)(\mu+1)+(\mu+k)} \right\} b_k \right| \\ &\geq |z_1^{\rho(\mu+1)} - z_2^{\rho(\mu+1)}| \left[(\mu+1) - \frac{\gamma}{\rho} - [\rho(\mu+1) + \zeta] |b_1| \right. \\ &\quad \left. - |z_2^{\rho(\mu+1)}| \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) (|a_k| + |b_k|) \right] \\ &\geq |z_1^{\rho(\mu+1)} - z_2^{\rho(\mu+1)}| \left[(\mu+1) - \frac{\gamma}{\rho} - [\rho(\mu+1) + \zeta] |b_1| \right] (1 - |z_2|^{\rho(\mu+1)}) > 0 \end{aligned}$$

Hence, $f(z_1)^\rho \neq f(z_2)^\rho$, which shows that f^ρ is univalent in U .

We prove that f^ρ is sense preserving in U , since by using (10), $|z| < 1$,

$$\begin{aligned} |h'(z)^\rho| &\geq \rho(\mu+1) - \sum_{k=2}^{\infty} \rho[(\mu+k) + (\rho-1)(\mu+1)] |a_k| \\ &> \rho(\mu+1) - \gamma - \sum_{k=2}^{\infty} \rho[(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |a_k| \\ &\geq \sum_{k=1}^{\infty} \rho[(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |b_k| \geq |g'(z)^\rho| \end{aligned}$$

We now show that $f(z)^\rho \in \Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)$. In (8), we make use of the fact that $Re\{w\} > \gamma$ if and only if $|1+w-\gamma| > |1-w+\gamma|$.

By substituting for $h(z)^\rho$ and $g(z)^\rho$ from (4) and using (10), we get

$$\begin{aligned} &\left| 1 + \zeta \tilde{D}_\lambda^{m,\alpha} \left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}} \right)} \right] + \tilde{D}_\lambda^{m,\alpha} [h'(z)^\rho + \overline{g'(z)^\rho}] - \zeta - \gamma \right| \\ &\quad - \left| 1 - \zeta \tilde{D}_\lambda^{m,\alpha} \left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}} \right)} \right] - \tilde{D}_\lambda^{m,\alpha} [h'(z)^\rho + \overline{g'(z)^\rho}] + \zeta + \gamma \right| \\ &= \left| 1 + \rho(\mu+1) z^{\rho(\mu+1)-1} + \rho[\zeta + \rho(\mu+1) \overline{z^{\rho(\mu+1)-1}}] \overline{b_1} - \gamma + \zeta \rho \sum_{k=2}^{\infty} \phi_{k,m}(\alpha, \lambda) (a_k z^{k-1} + \overline{b_k z^{k-1}}) \right| \end{aligned}$$

$$\begin{aligned}
& + \rho \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1)] \phi_{k,m}(\alpha, \lambda) (a_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1} + \overline{b_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}}) \\
& - \left| 1 - \rho(\mu+1)z^{\rho(\mu+1)-1} - \rho[\zeta + \rho(\mu+1)\overline{z^{\rho(\mu+1)-1}}]\overline{b_1} + \gamma - \zeta \rho \sum_{k=2}^{\infty} \phi_{k,m}(\alpha, \lambda) (a_k z^{k-1} + \overline{b_k z^{k-1}}) \right. \\
& \left. - \rho \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1)] \phi_{k,m}(\alpha, \lambda) (a_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1} + \overline{b_k z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}}) \right| \\
& \geq 2\rho(\mu+1) - 2\gamma - 2\rho[\zeta + \rho(\mu+1)]|b_1| \\
& - 2\rho \sum_{k=2}^{\infty} [\zeta + (\mu+k) + (\rho-1)(\mu+1)] \phi_{k,m}(\alpha, \lambda) (|a_k| + |b_k|) |z|^{[(\mu+k)+(\rho-1)(\mu+1)]-1} \\
& > 2 \left[\rho(\mu+1) + \rho[\zeta + \rho(\mu+1)] - \gamma - \rho \sum_{k=1}^{\infty} [\zeta + (\mu+k) + (\rho-1)(\mu+1)] \phi_{k,m}(\alpha, \lambda) (|a_k| + |b_k|) \right] \geq 0
\end{aligned}$$

The coefficient bound given by (10) can not be improved upon. The harmonic function given by

$$\begin{aligned}
f(z)^\rho &= z^{\rho(\mu+1)} + \sum_{k=2}^{\infty} \frac{(1 - \frac{\gamma}{\rho})x_k}{[(\rho-1)(\mu+1) + (\mu+k) + \zeta]} z^{(\rho-1)(\mu+1) + (\mu+k)} \\
&+ \sum_{k=1}^{\infty} \frac{(1 - \frac{\gamma}{\rho})\overline{y_k}}{[(\rho-1)(\mu+1) + (\mu+k) + \zeta]} \overline{z^{(\rho-1)(\mu+1) + (\mu+k)}}
\end{aligned}$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, $\rho > 0$, $\zeta \geq 0$, $0 \leq \mu < 1$ and $0 \leq \gamma < 1$ yields the extremal.

Corollary 3.4. *Let $f(z)^1 = f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are given by (8). Furthermore, let*

$$\sum_{k=1}^{\infty} (k + \zeta) \phi_{k,m}(\alpha, \lambda) [|a_k| + |b_k|] \leq 2 + \zeta - \gamma, \quad |b_1| < \frac{1 - \gamma}{1 + \zeta} < 1$$

where $a_1 = 1$, $\zeta \geq 0$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$, $\lambda \geq 0$ and $z \in U$. Then $f(z)$ is harmonic univalent and sense preserving in U and $f(z) \in \Psi_\gamma^{1,0}(m, \alpha, \zeta, \lambda)$.

Proof. Put $\rho = 1$ and $\mu = 0$ in Theorem 3.1.

Remark. This is the result obtained by Ezhilarasi *et al.* (2014).

Corollary 3.5. Let $f(z)^1 = f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are given by (8). Furthermore, let

$$\sum_{k=1}^{\infty} k\phi_{k,m}(\alpha, \lambda)[|a_k| + |b_k|] \leq 2 - \gamma, \quad |b_1| < 1 - \gamma < 1$$

where $a_1 = 1$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$, $\lambda \geq 0$ and $z \in U$.

Then $f(z)$ is harmonic univalent and sense preserving in U and $f(z) \in \Psi_{\gamma}^{1,0}(m, \alpha, 0, \lambda)$.

Proof. Put $\rho = 1$ and $\zeta = \mu = 0$ in Theorem 3.1.

Remark. This is the result obtained by Noor *et al.* (2009).

We also prove that the coefficient condition given in (10) is necessary and sufficient condition for $f^{\rho} \in \overline{\Psi_{\gamma}^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$.

Theorem 3.6. Let $f(z)^{\rho} = h(z)^{\rho} + \overline{g(z)^{\rho}}$, where $h(z)^{\rho}$, $g(z)^{\rho}$ are given by (9) then $f^{\rho} \in \overline{\Psi_{\gamma}^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$ if and only if

$$\sum_{k=1}^{\infty} [(\mu + k) + (\rho - 1)(\mu + 1) + \zeta] \phi_{k,m}(\alpha, \lambda)(|a_k| + |b_k|) \leq (\rho + 1)(\mu + 1) + \zeta - \frac{\gamma}{\rho}$$

where $a_1 = 1$, $\rho > 0$, $\zeta \geq 0$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$.

Proof. Suppose that $f^{\rho} \in \overline{\Psi_{\gamma}^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$. Then we find from (8) with $h(z)^{\rho}$, $g(z)^{\rho}$ given by (9) that

$$\begin{aligned} & Re \left\{ \zeta \tilde{D}_{\lambda}^{m,\alpha} \left[\frac{h(z)^{\rho}}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^{\rho}}{z^{\rho(\mu+1)}} \right)} \right] + \tilde{D}_{\lambda}^{m,\alpha} \left[h'(z)^{\rho} + \overline{g'(z)^{\rho}} \right] - \zeta \right\} \\ &= Re \left\{ \rho(\mu + 1)z^{\rho(\mu+1)-1} - \rho[\zeta + \rho(\mu + 1)z^{\rho(\mu+1)-1}]|b_1| - \rho \sum_{k=2}^{\infty} [\zeta(|a_k|z^{k-1} + |b_k|\overline{z^{k-1}}) \right. \\ &\quad \left. + [(\mu + k) + (\rho - 1)(\mu + 1)](|a_k|z^{[(\mu+k)+(\rho-1)(\mu+1)]-1} + |b_k|\overline{z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}})]\phi_{k,m}(\alpha, \lambda) \right\} > \gamma \end{aligned}$$

$\rho > 0$, $\zeta \geq 0$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$ and $0 \leq \alpha < 1$

If we choose z to be real and let $z \rightarrow 1^{-1}$, we get

$$\rho(\mu+1) - \rho[\zeta + \rho(\mu+1)]|b_1| - \rho \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda)(|a_k| + |b_k|) \geq \gamma$$

which yields the assertion (10) of Theorem 3.1.

Conversely, assume that (10) of Theorem 3.1 holds. Then we find from (8) with $h(z)^\rho$ and $g(z)^\rho$ given by (9) that on using (10), for $|z| < 1$

$$\begin{aligned} & \operatorname{Re} \left\{ \zeta \tilde{D}_\lambda^{m,\alpha} \left[\frac{h(z)^\rho}{z^{\rho(\mu+1)}} + \overline{\left(\frac{g(z)^\rho}{z^{\rho(\mu+1)}} \right)} \right] + \tilde{D}_\lambda^{m,\alpha} \left[h'(z)^\rho + \overline{g'(z)^\rho} \right] - \zeta \right\} \\ &= \operatorname{Re} \left\{ \rho(\mu+1)z^{\rho(\mu+1)-1} - \rho[\zeta + \rho(\mu+1)\overline{z^{\rho(\mu+1)-1}}]|b_1| - \rho \sum_{k=2}^{\infty} [\zeta(|a_k|z^{k-1} + |b_k|\overline{z^{k-1}}) \right. \\ & \quad \left. + [(\mu+k) + (\rho-1)(\mu+1)](|a_k|z^{[(\mu+k)+(\rho-1)(\mu+1)]-1} + |b_k|\overline{z^{[(\mu+k)+(\rho-1)(\mu+1)]-1}})]\phi_{k,m}(\alpha, \lambda) \right\} \\ &\geq \rho[(\rho+1)(\mu+1)+\zeta] - \rho \sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)(|a_k| + |b_k|)z^{[(\mu+k)+(\rho-1)(\mu+1)]-1} > \gamma \end{aligned}$$

Which yields that $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$.

In the next theorem, we state and prove covering theorem for functions $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$.

Theorem 3.7. *Let $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$. Then we have that*

$$\begin{aligned} & f(z)^\rho \leq (1 + \rho|b_1|)r^{\rho(\mu+1)} \\ & + \frac{\rho(2-\alpha)^m}{[(\mu+2) + (\rho-1)(\mu+1) + \zeta]2^m(1+\lambda)^m} \left[(\mu+1) - \frac{\gamma}{\rho} - [\rho(\mu+1) + \zeta]|b_1| \right] r^{(\rho-1)(\mu+1)+(\mu+2)} \end{aligned}$$

and

$$\begin{aligned} & f(z)^\rho \geq (1 - \rho|b_1|)r^{\rho(\mu+1)} \\ & - \frac{\rho(2-\alpha)^m}{[(\mu+2) + (\rho-1)(\mu+1) + \zeta]2^m(1+\lambda)^m} \left[(\mu+1) - \frac{\gamma}{\rho} - [\rho(\mu+1) + \zeta]|b_1| \right] r^{(\rho-1)(\mu+1)+(\mu+2)} \end{aligned}$$

for $|z| = r < 1$,

Proof. Let $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$, where $h(z)^\rho$, $g(z)^\rho$ are given by (10). Then

$$\begin{aligned} & |f(z)^\rho| = \left| z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho|a_k|z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho|b_k|z^{(\rho-1)(\mu+1)+(\mu+k)} \right| \\ &\geq (1 - \rho|b_1|)r^{\rho(\mu+1)} \\ & - \frac{\rho(2-\alpha)^m}{[(\mu+2) + (\rho-1)(\mu+1) + \zeta]2^m(1+\lambda)^m} \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(|a_k| + |b_k|)r^{(\rho-1)(\mu+1)+(\mu+2)} \\ &\geq (1 - \rho|b_1|)r^{\rho(\mu+1)} \\ & - \frac{\rho(2-\alpha)^m}{[(\mu+2) + (\rho-1)(\mu+1) + \zeta]2^m(1+\lambda)^m} \left[(\mu+1) - \frac{\gamma}{\rho} - [\rho(\mu+1) + \zeta]|b_1| \right] r^{(\rho-1)(\mu+1)+(\mu+2)} \end{aligned}$$

The proof of the upper bounds of $f(z)^\rho$ is similar.

Theorem 3.8. *Let $f(z)^\rho = h(z)^\rho + \overline{g(z)^\rho}$, where $h(z)^\rho, g(z)^\rho$ are given by (9). Then $f^\rho \in \overline{\Psi}_\gamma^{\rho, \mu}(m, \alpha, \zeta, \lambda)$ if and only if*

$$f(z)^\rho = \sum_{k=1}^{\infty} (\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho),$$

where

$$h_1(z)^\rho = z^{\rho(\mu+1)}$$

$$h_k(z)^\rho = z^{\rho(\mu+1)} - \frac{(1 - \frac{\gamma}{\rho})}{[(\mu + k) + (\rho - 1)(\mu + 1) + \zeta] \phi_{k,m}(\alpha, \lambda)} z^{(\rho-1)(\mu+1)+(\mu+k)} \quad k = 2, 3, \dots$$

$$g_k(z)^\rho = z^{\rho(\mu+1)} - \frac{(1 - \frac{\gamma}{\rho})}{[(\mu + k) + (\rho - 1)(\mu + 1) + \zeta] \phi_{k,m}(\alpha, \lambda)} \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}} \quad k = 1, 2, \dots$$

and

$$\sum_{k=1}^{\infty} (\lambda_k + \mu_k) = 1 \quad \lambda_k \geq 0, \mu_k \geq 0$$

Proof. Suppose $f(z)^\rho = \sum_{k=1}^{\infty} (\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho)$

$$\begin{aligned} &= \lambda_1 h_1(z)^\rho + \mu_1 g_1(z)^\rho + \sum_{k=2}^{\infty} (\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho) \\ &= \sum_{k=1}^{\infty} (\lambda_k + \mu_k) z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \frac{(1 - \frac{\gamma}{\rho}) \lambda_k}{[(\mu + k) + (\rho - 1)(\mu + 1) + \zeta] \phi_{k,m}(\alpha, \lambda)} z^{(\rho-1)(\mu+1)+(\mu+k)} \\ &\quad - \sum_{k=1}^{\infty} \frac{(1 - \frac{\gamma}{\rho}) \mu_k}{[(\mu + k) + (\rho - 1)(\mu + 1) + \zeta] \phi_{k,m}(\alpha, \lambda)} \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}} \\ &= z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho |a_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho |b_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}} \end{aligned}$$

where

$$\begin{aligned} a_1 &= 1 = \sum_{k=1}^{\infty} (\lambda_k + \mu_k) \\ |a_k| &= \frac{(1 - \frac{\gamma}{\rho}) \lambda_k}{\rho [(\mu + k) + (\rho - 1)(\mu + 1) + \zeta] \phi_{k,m}(\alpha, \lambda)}, \quad k \geq 2 \\ |b_k| &= \frac{(1 - \frac{\gamma}{\rho}) \mu_k}{\rho [(\mu + k) + (\rho - 1)(\mu + 1) + \zeta] \phi_{k,m}(\alpha, \lambda)}, \quad k \geq 1 \end{aligned}$$

Since,

$$\begin{aligned}
& \sum_{k=2}^{\infty} \frac{[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)}{1 - \frac{\gamma}{\rho}} |a_k| + \sum_{k=2}^{\infty} \frac{[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)}{1 - \frac{\gamma}{\rho}} |b_k| \\
&= \sum_{k=2}^{\infty} \frac{[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)}{1 - \frac{\gamma}{\rho}} \left(\frac{1 - \frac{\gamma}{\rho}}{\rho[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)} \right) \lambda_k \\
&+ \sum_{k=1}^{\infty} \frac{[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)}{1 - \frac{\gamma}{\rho}} \left(\frac{1 - \frac{\gamma}{\rho}}{\rho[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)} \right) \mu_k \\
&= \sum_{k=2}^{\infty} \frac{\lambda_k}{\rho} + \sum_{k=1}^{\infty} \frac{\mu_k}{\rho} = \frac{1}{\rho}(1 - \lambda_1) \leq 1
\end{aligned}$$

we have

$$\begin{aligned}
& \sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)(|a_k| + |b_k|) = \rho(\mu+1) + \zeta + (1 - \frac{\gamma}{\rho}) \left[\sum_{k=2}^{\infty} \frac{\lambda_k}{\rho} + \sum_{k=1}^{\infty} \frac{\mu_k}{\rho} \right] \\
& \leq \rho(\mu+1) + \zeta + 1 - \frac{\gamma}{\rho} = (\rho+1)(\mu+1) + \zeta + 1 - \frac{\gamma}{\rho} - \mu - 1 \leq (\rho+1)(\mu+1) + \zeta + 1 - \frac{\gamma}{\rho}
\end{aligned}$$

and so $f^\rho \in \overline{\Psi_\gamma^{\rho, \mu}(m, \alpha, \zeta, \lambda)}$.

Conversely, suppose $f^\rho \in \overline{\Psi_\gamma^{\rho, \mu}(m, \alpha, \zeta, \lambda)}$

Let

$$\begin{aligned}
\lambda_k &= \frac{\rho[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)|a_k|}{1 - \frac{\gamma}{\rho}}, \quad k = 2, 3, \dots \\
\mu_k &= \frac{\rho[(\mu+k) + (\rho-1)(\mu+1) + \zeta]\phi_{k,m}(\alpha, \lambda)|b_k|}{1 - \frac{\gamma}{\rho}}, \quad k = 1, 2, \dots \\
\lambda_1 &= 1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \mu_k
\end{aligned}$$

Now,

$$\begin{aligned}
f(z)^\rho &= z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho|a_k|z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho|b_k|\overline{z^{(\rho-1)(\mu+1)+(\mu+k)}} \\
&= z^{\rho(\mu+1)} + \sum_{k=2}^{\infty} \lambda_k \left(h_k(z)^\rho - z^{\rho(\mu+1)} \right) + \sum_{k=1}^{\infty} \mu_k \left(g_k(z)^\rho - z^{\rho(\mu+1)} \right) \\
&= \sum_{k=2}^{\infty} \lambda_k h_k(z)^\rho + \sum_{k=1}^{\infty} \mu_k g_k(z)^\rho + z^{\rho(\mu+1)} \left[1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \mu_k \right]
\end{aligned}$$

$$\implies f(z)^\rho = \sum_{k=1}^{\infty} (\lambda_k h_k(z)^\rho + \mu_k g_k(z)^\rho)$$

as required.

In particular the extreme points of $\overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$ are $\{h_k\}$ and $\{g_k\}$.

3.1. Convolution and Convex Combination. We now show that the class $\overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$ is invariant under convolution and convex combination of its elements.

For harmonic functions

$$f(z)^\rho = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho |a_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho |b_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

and

$$g(z)^\rho = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho |c_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho |d_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

The convolution (or Hadamard product) of f^ρ and g^ρ is given by (11)

$$(f^\rho * g^\rho)(z) = f(z)^\rho * g(z)^\rho = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho |a_k| |c_k| z^{(\rho-1)(\mu+1)+(\mu+k)} - \sum_{k=1}^{\infty} \rho |b_k| |d_k| \overline{z^{(\rho-1)(\mu+1)+(\mu+k)}}$$

Theorem 3.9. For $0 \leq \delta \leq \gamma < 1$, let $f^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$ and $g^\rho \in \overline{\Psi_\delta^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$. Then $f(z)^\rho * g(z)^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)} \subset \overline{\Psi_\delta^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$

Proof. The convolution $f(z)^\rho * g(z)^\rho$ is given by (11). We want to show that the coefficients of $f(z)^\rho * g(z)^\rho$ satisfy the condition given in (10).

For $g^\rho \in \overline{\Psi_\gamma^{\rho,\mu}(m, \alpha, \zeta, \lambda)}$, we note $|c_k| \leq 1$, $|d_k| \leq 1$.

Now for the convolution $f(z)^\rho * g(z)^\rho$, we obtain

$$\begin{aligned} & \sum_{k=1}^{\infty} [(\mu+k)+(\rho-1)(\mu+1)+\zeta] \phi_{k,m}(\alpha, \lambda) |a_k| |c_k| + \sum_{k=1}^{\infty} [(\mu+k)+(\rho-1)(\mu+1)+\zeta] \phi_{k,m}(\alpha, \lambda) |b_k| |d_k| \\ &= [\rho(\mu+1) + \zeta] |a_1| |c_1| + \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |a_k| |c_k| \\ & \quad + \sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |b_k| |d_k| \end{aligned}$$

$$\begin{aligned}
&\leq \rho(\mu+1) + \zeta + \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |a_k| + \sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |b_k| \\
&\leq (\rho+1)(\mu+1) + \zeta - \frac{\gamma}{\rho} \leq (\rho+1)(\mu+1) + \zeta - \frac{\delta}{\rho}
\end{aligned}$$

Thus the proof.

We now investigate the convex combination of members of $\overline{\Psi_{\gamma}^{\rho, \mu}(m, \alpha, \zeta, \lambda)}$.

Let the functions $f_i(z)^{\rho} \in \overline{\Psi_{\gamma}^{\rho, \mu}(m, \alpha, \zeta, \lambda)}$, where f_i^{ρ} , for $i = 1, 2, \dots, n$, is given by

$$(12) \quad f_i(z)^{\rho} = z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho |a_{k,i}| z^{(\rho-1)(\mu+1) + (\mu+k)} - \sum_{k=1}^{\infty} \rho |b_{k,i}| \overline{z^{(\rho-1)(\mu+1) + (\mu+k)}}$$

Theorem 3.10. *Let the functions $f_i(z)^{\rho}$ defined by (12) be in the class $\overline{\Psi_{\gamma}^{\rho, \mu}(m, \alpha, \zeta, \lambda)}$ for every $i = 1, 2, \dots, n$. Then the function $t_i(z)^{\rho}$ defined by*

$$t_i(z)^{\rho} = \sum_{i=1}^n c_i f_i(z)^{\rho} \quad (0 \leq c_i \leq 1)$$

is also in the class $\overline{\Psi_{\gamma}^{\rho, \mu}(m, \alpha, \zeta, \lambda)}$, where $\sum_{i=1}^n c_i = 1$

Proof. According to the definition of $t_i(z)^{\rho}$, we can write

$$\begin{aligned}
t_i(z)^{\rho} &= \sum_{i=1}^n c_i f_i(z)^{\rho} \\
&= \sum_{i=1}^n c_i \left[z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho |a_{k,i}| z^{(\rho-1)(\mu+1) + (\mu+k)} - \sum_{k=1}^{\infty} \rho |b_{k,i}| \overline{z^{(\rho-1)(\mu+1) + (\mu+k)}} \right] \\
&= z^{\rho(\mu+1)} - \sum_{k=2}^{\infty} \rho \left(\sum_{i=1}^n c_i |a_{k,i}| \right) z^{(\rho-1)(\mu+1) + (\mu+k)} - \sum_{k=1}^{\infty} \rho \left(\sum_{i=1}^n c_i |b_{k,i}| \right) \overline{z^{(\rho-1)(\mu+1) + (\mu+k)}}
\end{aligned}$$

Using (10), we have

$$\begin{aligned}
&[\rho(\mu+1) + \zeta] + \sum_{k=2}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) \left(\sum_{i=1}^n c_i |a_{k,i}| \right) \\
&\quad + \sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) \left(\sum_{i=1}^n c_i |b_{k,i}| \right) \\
&= \sum_{i=1}^n c_i \left(\sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |a_{k,i}| \right. \\
&\quad \left. + \sum_{k=1}^{\infty} [(\mu+k) + (\rho-1)(\mu+1) + \zeta] \phi_{k,m}(\alpha, \lambda) |b_{k,i}| \right) \leq (\rho+1)(\mu+1) + \zeta - \frac{\gamma}{\rho}
\end{aligned}$$

since each f_i^{ρ} is in $\overline{\Psi_{\gamma}^{\rho, \mu}(m, \alpha, \zeta, \lambda)}$. This completes the proof.

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